Série de Fourier

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Para uma função f periódica de período 2l, definida entre os intervalos $I = [-l, l]$, temos os coeficientes.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \; ; \; a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \; e \; b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Exemplo:

Condição $f(x) = f(x+2\pi)$, Determine a série de Fourier

- Nesta condição $f(x+2\pi)$, analisamos que $2\pi = 2l \therefore l = \pi$

a) $f(x) = \begin{cases} 0, \text{ se } & -\pi < x \leq 0 \\ \pi, \text{ se } & 0 < x < \pi \end{cases}$

1-) Determinar a_0

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \pi\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \pi(\pi - 0)\right] = \pi \rightarrow \boxed{a_0 = \pi}$$

2-) Determinar a_n $1+\pi$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(nx)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$a_n = \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0\cos(nx)\,dx}^{\,0} + \int_{0}^{\pi} \pi\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi\cdot\left(-\frac{1}{n}\right)\sin(nx)\Big|_{0}^{\pi}\right] = -\frac{1}{n}\left(\cancel{\sin(n\pi)}^{\,0} - \cancel{\sin(n\cdot 0)}^{\,0}\right)$$

$$\therefore \boxed{a_n = 0}$$

3-) Determinar b_n

$$\frac{d(\operatorname{sen}(x))}{dx} = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\operatorname{sen}(x)$$

$$\int \operatorname{sen}(x)dx = -\cos(x)$$

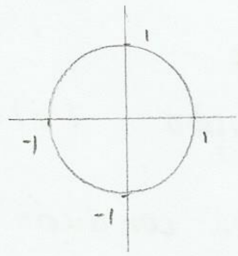
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\operatorname{sen}(n\pi x)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen}(nx)\,dx$$

$$= \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0\operatorname{sen}(nx)\,dx}^{\to 0} + \int_{0}^{\pi}\pi\operatorname{sen}(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos(nx)\Big|_0^{\pi}\right] = \left(-\frac{1}{n}\right)\left[\cos(n\pi) - \cos(n.0)\right]$$

$$= \frac{1}{n} - \frac{1}{n}\cos(n\pi)$$



Com isso temos que:

$$b_n = \begin{cases} 0, & \text{se } n \text{ par} \\ \frac{2}{n}, & \text{se } n \text{ ímpar} \end{cases}$$

4-) Série de Fourier. Com $\begin{cases} a_0 = \pi \\ a_n = 0 \\ b_n = \begin{cases} 0, \text{ se } n \text{ par} \\ \frac{2}{n}, \text{ se } n \text{ ímpar} \end{cases} \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty}\left(\cancel{a_n\cos(nx)}^{\to 0} + b_n\operatorname{sen}(nx)\right)$$

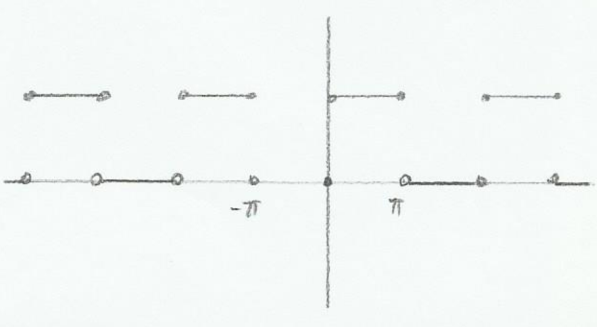
$$= \frac{a_0}{2} + \cancel{a_1\cos(nx)}^{\to 0} + b_1\operatorname{sen}(nx) + \cancel{a_2\cos(nx)} + \cancel{b_2\operatorname{sen}(nx)}^{\to 0} + \cancel{a_3\cos(nx)}^{\to 0} + \dots$$

$$= \frac{a_0}{2} + b_1\operatorname{sen}(nx) + b_3\operatorname{sen}(nx) + b_5\operatorname{sen}(nx) + \dots$$

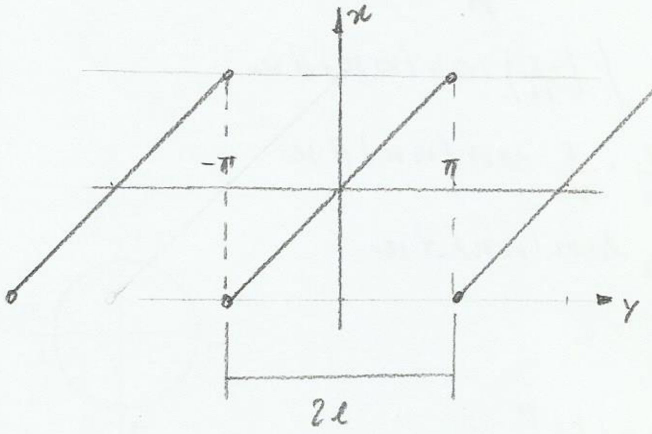
$$= \frac{\pi}{2} + \frac{2}{1}\operatorname{sen}(1x) + \frac{2}{3}\operatorname{sen}(3x) + \frac{2}{5}\operatorname{sen}(5x) + \dots$$

$$(1,3,5,\dots) \Rightarrow b_n = b_0 + (n-1)r \rightarrow b_n = 1 + (n-1).2$$
$$= 1 + 2n - 2 = 2n - 1$$

$$f(x) = \frac{\pi}{2} + \sum_{1}^{\infty}\frac{2}{(2n-1)}\operatorname{sen}[(2n-1)x]$$



$* \quad F(x) = x \quad ; \quad -\pi < x < \pi$



Como $2l = 2\pi$

$\therefore \ l = \pi$

1º) Determinar a_0

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\cdot x\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\cdot\frac{x^2}{2}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\cdot\frac{\pi^2}{2} - \frac{(-\pi)^2}{2} = 0$$

2º) Determinar a_n

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(nx\pi)\,dx \rightarrow a_n = \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}_{I}$$

$$I = \int_{-\pi}^{\pi} x\cos(nx)\,dx$$

$u = x \qquad dv = \cos(nx)\,dx$

$du = dx \qquad v = \frac{1}{n}\operatorname{sen}(nx)$

$$I = \int u\,dv = \frac{x}{n}\operatorname{sen}(nx) - \int\frac{1}{n}\operatorname{sen}(nx)\,dx$$

$$= \frac{1}{n}\left[x\operatorname{sen}(nx) + \frac{1}{n}\cos(nx)\right]$$

Para $n =$

$$a_n = \frac{1}{\pi}\cdot\frac{1}{n}\left[x\operatorname{sen}(nx) + \frac{1}{n}\cos(nx)\right]\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\cdot\frac{1}{n}\left[x(\underbrace{\cancel{\operatorname{sen}(\pi n)}}_{0} - \underbrace{\cancel{\operatorname{sen}(-\pi n)}}_{0}) + \frac{1}{n}(\underbrace{\cos(n\pi)}_{-1} - \underbrace{\cos(-\pi n)}_{-1})\right] \overset{0}{}$$

$\therefore \quad a_n = 0$

3º) Determinar b_n

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\operatorname{sen}(n\pi x)\,dx \rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen}(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\operatorname{sen}(nx)\,dx$$

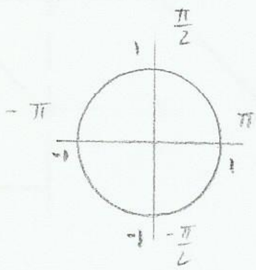
$I = \int x \, sen(nx) \, dx \qquad x = u \qquad dv = sen(nx)dx$

$\qquad\qquad dx = du \qquad v = \frac{-1}{n} \cos(nx)$

$I = \int u \, dv = x \cdot \left(\frac{-1}{n}\right) \cos(nx) - \int \left(\frac{-1}{n}\right) \cos(nx) dx$

$= x\left(\frac{-1}{n}\right) \cos(nx) + \frac{1}{n} \cdot \frac{1}{n} sen(nx) dx$

$= \frac{-x}{n} \cos(nx) + \frac{1}{n^2} sen(nx) dx$



$bn = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, sen(nx) dx$

$= \frac{1}{\pi} \cdot \left(\frac{-x}{n} \cos(nx) + \frac{1}{n^2} sen(nx) \right) \Big|_{-\pi}^{\pi}$

$= \frac{1}{\pi} \cdot \left[\frac{-\pi}{n} \left(\underbrace{\cos(n\pi)}_{1} - \underbrace{\cos(-\pi n)}_{1} \right) + \frac{1}{n^2} \underbrace{\left(sen(\pi n) - sen(-\pi n) \right)}_{0} \right]$

"Obs.: Como $\cos(-x) = -\cos(x)$"

$= \frac{1}{\pi} \left[-\frac{\pi}{n} \left(\cos(n\pi) - [-\cos(\pi n)] \right) \right]$

$= \frac{1}{\pi} \cdot \left[\frac{-\pi}{n} \cdot 2\cos(n\pi) \right] = \frac{-2}{n} \cos(n\pi)$

$\therefore bn = \begin{cases} \frac{-2}{n}, & se\ n\ par \\ \frac{2}{n}, & se\ n\ ímpar \end{cases}$

4º) Série de Fourier

$f(x) = \cancel{\frac{a_0}{2}}^{\to 0} + \sum_{1}^{\infty} \left(\cancel{an \cos(nx)}^{\to 0} + bn \, sen(nx) \right)$

$= b_1 sen(nx) + b_2 sen(nx) + b_3 sen(nx) + b_4 sen(nx) + \ldots$

$= \frac{2}{1} sen(1x) - \frac{2}{2} sen(2x) + \frac{2}{3} sen(3x) - \frac{2}{4} sen(4x) + \ldots$

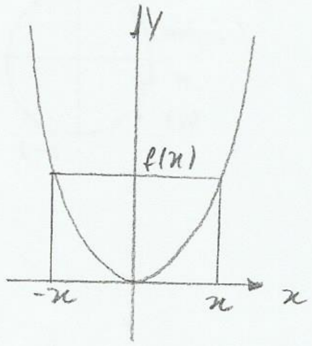
$f(x) = \sum_{1}^{\infty} \frac{2}{n} sen(nx) \cdot (-1)^{n+1}$

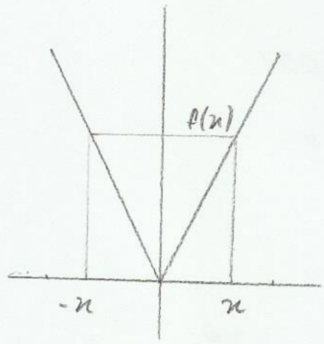
Série de Fourier incompleta

Funções pares e ímpares

- Uma função $f(x)$ definida em um intervalo simétrico $I =]-l, +l[$ é **par** se $f(x) = f(-x)$, para todo o x real. O seu gráfico é simétrico em relação ao eixo das ordenadas (eixo y).

Exemplos:





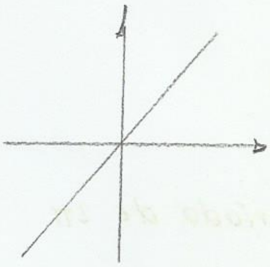
$f(-x) = f(x)$ $\qquad$ $f(-x) = f(x)$

Resumo

$$a_0 = 2\int_0^{l} f(x)\,dx$$
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$
$$b_n = 0$$

- Uma função $f(x)$ definida em um intervalo simétrico $I =]-l, +l[$ é **ímpar** se $f(x) = -f(x)$, para todo x real. O seu gráfico é simétrico em relação a origem.

Exemplos:



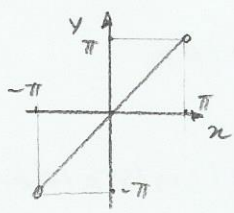
Resumo

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

* Determine a expansão da série de Fourier de $f(x) = x$, $-\pi < x < x$, com período igual a 2π

$2\pi = 2l \quad \therefore \quad l = \pi$

1º) Gráfico



f é ímpar, pois o gráfico é simétrico em relação a origem. ($f(x) = -f(x)$)

$\therefore$ Concluimos que

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\text{sen}(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,\text{sen}(nx)\,dx$$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\,\text{sen}(nx)\right]\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\cdot\left[\frac{1}{n}\left(-\pi\cos(n\pi) + \cancel{0\cos(0n)}^{\,0}\right) + \frac{1}{n^2}\left(\cancel{\text{sen}(n\pi)}^{\,0} - \cancel{\text{sen}(n0)}^{\,0}\right)\right]$$

$$= \frac{-2}{n}\cos(n\pi)$$

$$b_n = \begin{cases} \dfrac{2}{n}, & \text{se } n \text{ ímpar} \\ -\dfrac{2}{n}, & \text{se } n \text{ par} \end{cases}$$

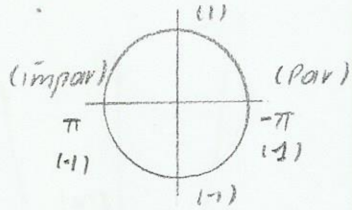
$I = \int x\,\text{sen}(nx)\,dx$

$u = x \qquad dv = \text{sen}(nx)\,dx$

$du = dx \qquad v = \frac{-1}{n}\cos(nx)$

$I = -\frac{x}{n}\cos(nx) - \int\left(\frac{-1}{n}\right)\cos(nx)\,dx$

$= -\frac{x}{n}\cos(nx) + \frac{1}{n}\cdot\frac{1}{n}\,\text{sen}(nx)$



- Série de Fourier

$$f(x) = \cancel{\frac{a_0}{2}}^{\,0} + \sum_1^{\infty}\left(\cancel{a_n\cos(nx)}^{\,0} + b_n\,\text{sen}(nx)\right)$$

$$= b_1\,\text{sen}(nx) + b_2\,\text{sen}(nx) + b_3\,\text{sen}(nx) + b_4\,\text{sen}(nx) + b_5\,\text{sen}(nx) + \ldots$$

$$= \frac{2}{1}\,\text{sen}(1x) - \frac{2}{2}\,\text{sen}(2x) + \frac{2}{3}\,\text{sen}(3x) + \frac{2}{4}\,\text{sen}(4x) - \frac{2}{5}\,\text{sen}(5x) + \ldots$$

$$f(x) = \sum_1^{\infty}\frac{2}{n}\,\text{sen}(nx)\,(-1)^{n+1}$$

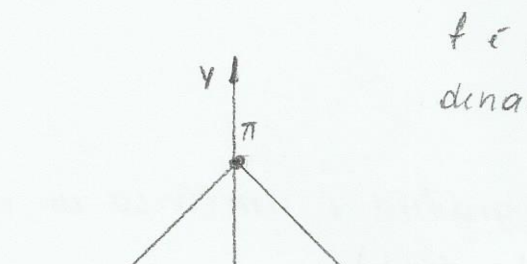
- Expandir a função

$$f(x) = \begin{cases} \pi + x, & \text{se } -\pi < x \le 0 \\ \pi - x, & \text{se } 0 < x < \pi \end{cases}, \text{ período de } 2\pi$$

$2\pi = 2l$

$\therefore \pi = l$

1º) Gráfico



f é par, pois é simétrico ao eixo das ordenadas (eixo y) e $f(x) = f(-x)$.

Portanto,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = 0$$

2-) $a_0 = ?$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{2}{\pi}\cdot\left(\pi x - \frac{x^2}{2}\right)\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\cdot\left[\pi(\pi - 0) - \frac{1}{2}(\pi^2 - 0)\right] = \frac{2}{\pi}\cdot\left(\pi^2 - \frac{1}{2}\pi^2\right) = \pi$$

3-) $a_n = ?$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos(nx)\,dx = \frac{2}{\pi}\cdot\Big[\underbrace{\int_0^{\pi}\pi\cos(nx)\,dx}_{I_1} - \underbrace{\int_0^{\pi}x\cos(nx)\,dx}_{I_2}\Big]$$

$$I_1 = \int_0^{\pi}\pi\cos(nx)\,dx = \frac{\pi}{n}\sin(nx)\Big|_0^{\pi} = \frac{\pi}{n}\cdot[\cancel{\sin(n\pi)}^{\to 0} - \cancel{\sin(n0)}^{\to 0}] = 0$$

$$I_2 = \int_0^{\pi} x\cos(nx)\,dx \qquad x = u \qquad dv = \cos(nx)\,dx$$

$$dx = du \qquad v = \frac{1}{n}\sin(nx)$$

$$= \frac{x}{n}\sin(nx) - \int\frac{1}{n}\sin(nx)\,dx$$

$$= \frac{x}{n}\sin(nx) - \frac{1}{n}\cdot\left(-\frac{1}{n}\right)\cos(nx)$$

$$= \frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}$$

$$= \frac{1}{n}(\pi\cancel{\sin(n\pi)}^{\to 0} - 0\cancel{\sin(n0)}^{\to 0}) + \frac{1}{n^2}(\cos(n\pi) - \cancel{\cos(n0)}^{1})$$

$$= \frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}$$

$$\therefore a_n = \frac{2}{\pi}\cdot\left[0 - \frac{1}{n^2}\cos(nx) + \frac{1}{n^2}\right] = \frac{2}{\pi}\left(\frac{1}{n^2} - \frac{1}{n^2}\cos(nx)\right)$$

$$a_n = \begin{cases}\frac{2}{\pi}\left(\frac{1}{n^2} + \frac{1}{n^2}\right), \text{ se } n \text{ impar}\\ \frac{2}{\pi}\left(\frac{1}{n^2} - \frac{1}{n^2}\right), \text{ se } n \text{ par}\end{cases} \sim a_n = \begin{cases}\frac{4}{\pi n^2}, \text{ se } n \text{ impar}\\ 0, \text{ se } n \text{ par}\end{cases}$$

Série de Fourier

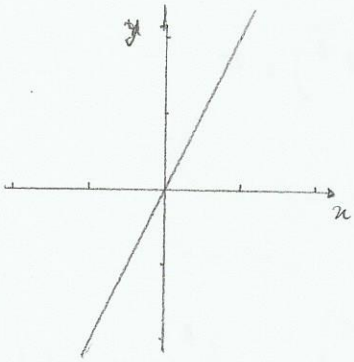
$$f(x) = \frac{a_0}{2} + \sum_1^{\infty}\left(a_n\cos(nx) + \cancel{b_n\sin(nx)}^{\to 0}\right)$$

$$= \frac{\pi}{2} + a_1\cos(1x) + \cancel{a_2\cos(2x)}^{\to 0} + a_3\cos(3x) + \cancel{a_4\cos(5x)}^{\to 0} + \cdots$$

$$= \frac{\pi}{2} + \frac{4}{\pi 1^2}\cos(1x) + \frac{4}{\pi\cdot 3^2}\cos(3x) + \frac{4}{\pi\cdot 5^2}\cos(5x) + \cdots$$

$$f(x) = \frac{\pi}{2} + \sum_{01}^{\infty} \frac{4}{\pi(2n-1)^2} \cos((2n-1)x)$$

2.) $f(x) = -2x; \quad -\pi < x < \pi$



função ímpar, pois o gráfico é simétrico a origem

com isso $\quad f(x) = -f(-x)$

$$a_n = 0$$
$$a_0 = 0$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}(-2x)\sin(nx)\,dx = \frac{-4}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$x = u \qquad dv = \sin(nx)\,dx$

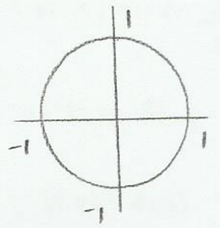
$dx = du \qquad v = -\frac{1}{n}\cos(nx)$

$$I = \frac{-x}{n}\sin(nx) + \frac{1}{n}\int \cos(nx)\,dx$$
$$= -\frac{x}{n}\cos(nx) + \frac{1}{n}\int \cos(nx)\,dx$$

$$b_n = \frac{-4}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n}\cdot\frac{1}{n^2}\sin(nx)\right]\Big|_0^{\pi}$$

$$= \frac{-4}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \cancel{\frac{1}{n^2}\sin(n\pi)}^{\,0} - \left(\cancel{\frac{-0}{n}\cos(n0)}^{\,0} + \cancel{\frac{1}{n^2}\sin(n0)}^{\,0}\right)\right]$$

$$= \frac{4}{n}\cos(n\pi)$$



$$= \begin{cases} \frac{-4}{n}, & \text{se } n \text{ ímpar} \\ \frac{4}{n}, & \text{se } n \text{ par} \end{cases}$$

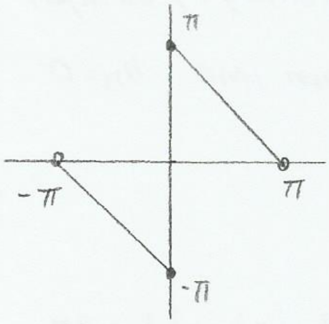
$$f(x) = \cancel{\frac{a_0}{2}}^{\,0} + \sum_{1}^{\infty}\left(\cancel{a_n\cos(nx)}^{\,0} + b_n\sin(nx)\right)$$

$$= b_1\sin(nx) + b_2\sin(nx) + b_3\sin(nx) + b_4\sin(nx) + b_5(\sin(nx)) + \dots$$

$$= -\frac{4}{1}\sin(1x) + \frac{4}{2}\sin(2x) - \frac{4}{3}\sin(3x) + \frac{4}{4}\sin(4x) + \frac{4}{5}\sin(5x) +$$

$$\therefore \quad f(x) = \sum_{1}^{\infty} \frac{4\sin(nx)}{n}\cdot(-1)^n$$

$$f(x) = \begin{cases} -\pi - x, & \text{se } -\pi < x \leq 0 \\ \pi - x, & \text{se } 0 < x < \pi \end{cases}$$



f é impar, pois o gráfico é simétrico a origem

com isso: $f(x) = -f(-x)$ e $a_0 = 0$ e $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\,sen(nx)\,dx = \frac{2}{\pi}\int_0^\pi (\pi - x)\,sen(nx)\,dx$$

$u = \pi - x \qquad dv = sen(nx)\,dx$

$du = -dx \qquad v = -\frac{1}{n}\cos(nx)$

$$\int(\pi - x)\,sen(nx)\,dx = -\frac{(\pi - x)}{n}\cos(nx) - \int\left(\frac{-1}{n}\right)\cos(nx)\,(-1)\,dx$$

$$= \frac{x - \pi}{n}\cos(nx) - \frac{1}{n}\int\cos(nx)\,dx$$

$$= \frac{(x - \pi)}{n}\cos(nx) - \frac{1}{n^2}\,sen(nx)\Big|_0^\pi$$

$$= \cancel{\frac{(\pi - \pi)}{n}\cos(n\pi)}^{\to 0} - \cancel{\frac{1}{n^2}\,sen(n\pi)}^{\to 0} - \frac{(0 - \pi)}{n}\cos(n0) - \cancel{\frac{1}{n^2}\,sen(0n)}^{\to 0}$$

$$= \frac{\pi}{n}$$

$$b_n = \frac{2}{\pi}\,\frac{\pi}{n} = \frac{2}{n}$$

Série de Fourier

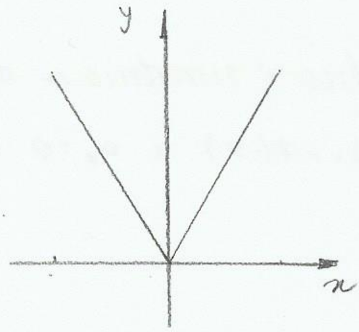
$$f(x) = \cancel{a_0}^{\to 0} + \sum_1^\infty \left(a_n \cancel{\cos(nx)}^{\to 0} + b_n\,sen(nx)\right)$$

$$= b_1\,sen(1x) + b_2\,sen(2x) + b_3\,sen(3x) + \dots$$

$$= \frac{2}{1}\,sen(1x) + \frac{2}{2}\,sen(2x) + \frac{2}{3}\,sen(3x) + \dots$$

$$\therefore f(x) = \sum_1^\infty \frac{2\,sen(nx)}{n}$$

$f(x) = |-2x|$



$f(x)$ é par, pois seu gráfico é simétrico ao eixo das ordenadas (eixo y), ou seja, $f(x) = f(-x)$, e com isso $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} 2x\,dx = \frac{2\cdot\cancel{2}}{\pi}\cdot\frac{x^2}{\cancel{2}}\Big|_0^{\pi} = \frac{2}{\pi}\left(\pi^2 - 0^2\right) = 2\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} 2x\cos(nx)\,dx$$

$$\int_0^{\pi} x\cos(nx)\,dx = x\,\frac{1}{n}\,\mathrm{sen}(nx) - \frac{1}{n}\int \mathrm{sen}(nx)\,dx$$

$u = x \qquad dv = \cos(nx)\,dx$

$du = dx \qquad v = \frac{1}{n}\,\mathrm{sen}(nx)$

$$\int_0^{\pi} x\cos(nx)\,dx = \frac{x}{n}\,\mathrm{sen}(nx) + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}$$

$$= \cancel{\frac{\pi}{n}\,\mathrm{sen}(\pi n)}^{\to 0} + \frac{1}{n^2}\cos(n\pi) - \cancel{\frac{0\,\mathrm{sen}(n0)}{n}}^{\to 0} - \frac{1}{n^2}\cos(n.0)$$

$$a_n = \frac{2}{\pi}\cdot 2\left(\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\right) = \begin{cases} 0, & \text{se } n \text{ par} \\ \dfrac{-8}{\pi n^2}, & \text{se } n \text{ ímpar} \end{cases}$$

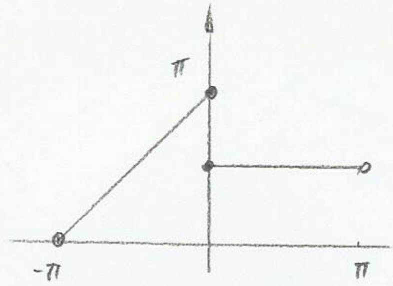
$$f(x) = \frac{a_0}{2} + \sum_1^{\infty}\left(a_n\cos(nx) + \cancel{b_n\,\mathrm{sen}(nx)}^{\to 0}\right)$$

$$= \frac{2\pi}{2} + a_1\cos(1x) + a_2\cancel{\cos(2x)}^{\to 0} + a_3\cos(3x) + a_4\cancel{\cos(4x)}^{\to 0} + \cdots$$

$$= \frac{2\pi}{2} - \frac{8}{\pi.1^2}\cos(1x) - \frac{8}{\pi.3^2}\cos(3x) - \frac{8}{\pi 5^2}\cos(5x) + .$$

$$\therefore\ f(x) = \pi - \sum_1^{\infty}\frac{8\cos[(2n-1)x]}{\pi\,(2n-1)^2}$$

$f(x): \begin{cases} x+\pi, & -\pi < x \leq 0 \\ \frac{\pi}{2}, & 0 < x < \pi \end{cases}$ Achar apenas a_0 e b_n



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx$$

$$= \frac{1}{\pi} \cdot \left[\int_{-\pi}^{0} (x+\pi) \sin(nx)\,dx + \int_{0}^{\pi} \frac{\pi}{2} \sin(nx)\,dx \right]$$

$$\int_{-\pi}^{0} (x+\pi) \sin(nx)\,dx =$$

$(x+\pi) = u$ $\quad dv = \sin(nx)\,dx$

$dx = du$ $\quad v = \frac{-1}{n} \cos(nx).$

$$= \frac{-(x+\pi)}{n} \cos(nx) + \frac{1}{n} \int \cos(nx)\,dx$$

$$= \frac{(\pi - x)}{n} \cos(nx) + \frac{1}{n^2} \sin(nx) \Big|_{-\pi}^{0}$$

$$= \frac{(\pi - 0)}{n} \cos(0n) + \cancelto{0}{\frac{1}{n^2} \sin(0n)} - \frac{(\pi+\pi)}{n} \cos(n\pi) \cancelto{0}{- \frac{1}{n^2} \sin(-\pi n)}$$

$$= \frac{1}{n} \left(\pi - 2\cos(n\pi) \right)$$

$$\int_{0}^{\pi} \frac{\pi}{2} \sin(nx)\,dx = \frac{\pi}{2} \cdot \left(\frac{-1}{n} \right) \cos(nx) \Big|_{0}^{\pi} = \frac{-\pi}{2n} \cos(n\pi) + \frac{\pi}{2n} \cos(0n)$$

$$= \frac{\pi}{2n} \left(1 - \cos(n\pi) \right)$$

$$a_n = \frac{1}{\pi} \cdot \left[\frac{1}{n} \left(\pi - 2\cos(n\pi) \right) + \frac{\pi}{2n} \left(1 - \cos(n\pi) \right) \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (x+\pi)\,dx + \int_{0}^{\pi} \frac{\pi}{2}\,dx \right]$$

$$= \frac{1}{\pi} \cdot \left[\frac{x^2}{2} + \pi x \Big|_{-\pi}^{0} + \frac{\pi}{2} x \Big|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{0}{2} + 0\pi \cancel{- \frac{\pi^2}{2}} + \pi^2 \cancel{+ \frac{\pi^2}{2}} \right. = \pi \quad \therefore \boxed{a_0 = \pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\sin(nx)\,dx + \int_{0}^{\pi}\frac{\pi}{2}\sin(nx)\,dx\right]$$

$$\int_{-\pi}^{0}(x+\pi)\sin(nx)\,dx$$

$(x+\pi) = u \qquad dv = \sin(nx)\,dx$

$dx = du \qquad v = -\frac{1}{n}\cos(nx)$

$$\int_{-\pi}^{0}(x+\pi)\sin(nx) = \frac{-(x+\pi)}{n}\cos(nx) + \frac{1}{n}\int\cos(nx)\,dx)$$

$$= \frac{\pi - x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\Big|_{-\pi}^{0}$$

$$= \frac{(\pi-0)}{n}\cos(0x) + \cancel{\frac{1}{n^2}\sin(\overset{\to 0}{0.x})} - \frac{(\pi+\pi)}{n}\cos(-\pi n) - \cancel{\frac{1}{n^2}\sin(n(-\pi))}^{\to 0}$$

$$= \frac{1}{n}\cdot(\pi - 2\pi\cos(-n\pi)) \quad ; \quad \cos(-x) = \cos(x)$$

$$= \frac{1}{n}(\pi - 2\pi\cos(n\pi))$$

$$\int_{0}^{\pi}\frac{\pi}{2}\sin(nx)\,dx = \frac{\pi}{2}\cdot\left(\frac{-1}{n}\right)\cos(nx)\Big|_{0}^{\pi}$$

$$= \frac{-\pi}{2n}(\cos(n\pi) - \cos(n0))$$

$$= \frac{-\pi}{2n}(\cos(n\pi) - 1)$$

$$b_n = \frac{1}{\pi}\left[\frac{1}{n}(\pi - 2\pi\cos(n\pi)) - \frac{\pi}{2n}(\cos(n\pi)-1)\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{n} - \frac{2\cancel{\pi}\cos(n\pi)}{n} - \frac{\cancel{\pi}\cos(n\pi)}{2n} + \frac{\cancel{\pi}}{2n}\right]$$

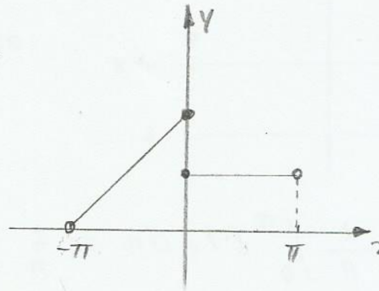
$$= \frac{-5\cos(n\pi)}{2n} + \frac{3}{2n} = \frac{-5\cos(n\pi)+3}{2n}$$

$$b_n = \begin{cases} -\frac{1}{n}, & \text{se } n \text{ par} \\ \frac{4}{n}, & \text{se } n \text{ impar} \end{cases}$$

Exercícios de Prova de Fourier

1-) Calcular a_0 e b_n

$$f(x) = \begin{cases} x + \pi, & \text{se } -\pi < x \le 0 \\ \frac{\pi}{2}, & \text{se } 0 < x < \pi \end{cases}$$



$2\pi = 2l$

$\therefore l = \pi$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (x+\pi)\,dx + \int_0^{\pi} \frac{\pi}{2}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \pi x \Big|_{-\pi}^{0} + \frac{\pi}{2}x\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{0^2}{2}} + \cancel{\pi 0} - \left(\frac{(-\pi)^2}{2} - \pi^2\right) + \frac{\pi^2}{2} - \cancel{\frac{\pi 0}{2}}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{2} + \pi^2 + \frac{\pi^2}{2}\right] = \pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{sen}(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (x+\pi)\,\text{sen}(nx)\,dx + \int_0^{\pi} \frac{\pi}{2}\,\text{sen}(nx)\,dx\right]$$

$$\int_{-\pi}^{0} (x+\pi)\,\text{sen}(nx)\,dx \qquad (x+\pi) = u \qquad dv = \text{sen}(nx)\,dx$$

$$dx = du \qquad v = -\frac{1}{n}\cos(nx)$$

$$= \frac{-(x+\pi)}{n}\cos(nx) + \frac{1}{n}\int \cos(nx)\,dx$$

$$= \frac{-(x+\pi)}{n}\cos(nx) + \frac{1}{n^2}\,\text{sen}(nx)\Big|_{-\pi}^{0}$$

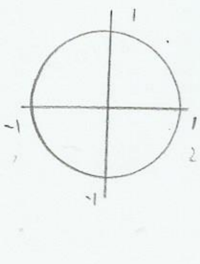
$$= \frac{-(0+\pi)}{n}\cos(n0) + \cancel{\frac{1}{n^2}\,\text{sen}(n.0)} - \left(\cancel{\frac{-(-\pi+\pi)}{n}}\cos(-\pi n) + \cancel{\frac{1}{n^2}\,\text{sen}(-\pi n)}\right)$$

$$= \frac{-\pi}{n}$$

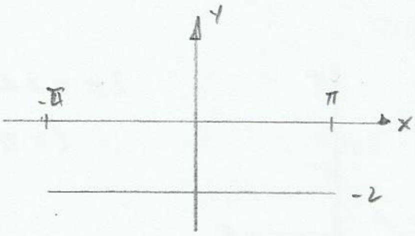
$$\int_0^{\pi} \frac{\pi}{2}\,\text{sen}(nx)\,dx = \frac{\pi}{2}\left(-\frac{1}{n}\right)\cos(nx)\Big|_0^{\pi} = \frac{-\pi}{2n}\left(\cos(n\pi) - \cos(n0)\right)$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi}{n} - \frac{\pi}{2n}(\cos(n\pi) - 1)\right]$$

$$b_n = \begin{cases} \frac{1}{\pi}\left[\frac{-\pi}{n} + \cancel{\frac{\pi}{2n}} - \cancel{\frac{\pi}{2n}}\right] = \frac{-1}{n} & \text{se } n \text{ par} \\ \frac{1}{\pi}\left[\frac{-\pi}{n} + \frac{\pi}{2n} + \frac{\pi}{2n}\right] = \frac{1}{\pi}\left[\frac{-2\pi + 2\pi}{2n}\right] = 0 & \text{se } n \text{ ímpar} \end{cases}$$



$f(x) = -2 \; ; \quad -\pi < x < \pi$



função par, pois o gráfico é simétrico ao eixo das ordenadas (eixo y)

$f(x) = f(-x) \quad \therefore \quad b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^\pi (-2)\,dx = \frac{2}{\pi}\cdot(-2)\cdot x\Big|_0^\pi = \frac{-4}{\pi}(\pi - 0) = -4$$

$$a_n = \frac{2}{\pi}\int_0^\pi (-2)\cos(nx)\,dx = \frac{-4}{\pi}\cdot\frac{1}{n}\,\mathrm{sen}(nx)\Big|_0^\pi$$

$$= \frac{-4}{n\pi}\cdot\left(\cancelto{0}{\mathrm{sen}(n\pi)} - \cancelto{0}{\mathrm{sen}(n0)}\right) = 0$$

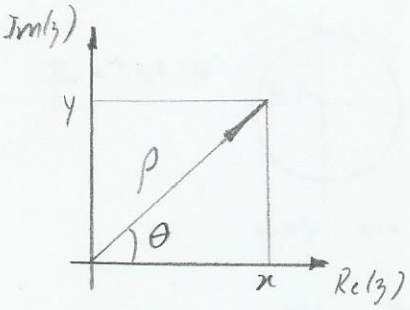
$$b_n = \frac{2}{\pi}\int_0^\pi (-2)\,\mathrm{sen}(nx)\,dx = \frac{+4}{\pi}\cdot\frac{1}{n}\cos(nx)\Big|_0^\pi = \frac{+4}{\pi n}\left(\cos(n\pi) - \cos(n0)\right)$$

$$= \frac{4}{\pi n}\cdot(\cos(n\pi) - 1)$$

$$= \frac{4}{\pi n}(1-1) = 0 \; , \text{se } n \text{ par}$$

$$= \frac{4}{\pi n}(-1-1) = \frac{-8}{\pi n} \; , \text{se } n \text{ ímpar}$$

Números complexos



ρ: módulo, sendo $\rho \geq 0$

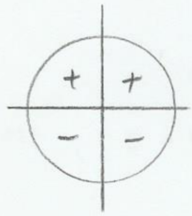
θ: argumento $0 \leq \theta \leq 2\pi$

$$\rho^2 = x^2 + y^2$$

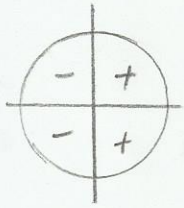
$$\cos\theta = \frac{y}{\rho} \qquad \operatorname{sen}\theta = \frac{x}{\rho}$$

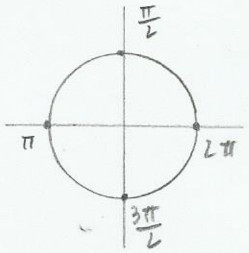
$\boxed{\text{Revisão trigonométrica}}$

sen α



cos α

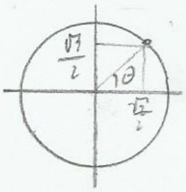




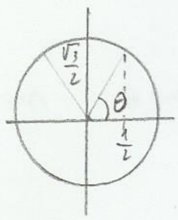
Exemplo: $(0,1)$

$$\rho^2 = 0^2 + 1^2 \quad \therefore \rho = 1$$

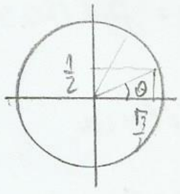
$$\operatorname{sen}\alpha = \frac{1}{1} \quad \cos\alpha = \frac{0}{1} \quad \therefore \alpha = \frac{\pi}{2}$$



$\theta = 45° = \frac{\pi}{4}$



$\theta = 60° = \frac{\pi}{3}$



$\theta = 30° = \frac{\pi}{6}$

Redução para o 1º quadrante

1º q: $\theta = \alpha$

2º q: $\theta = \pi - \alpha$

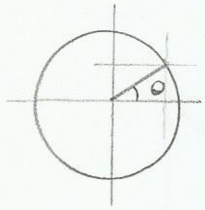
3º q: $\theta = \pi + \alpha$

4º q: $\theta = 2\pi - \alpha$

Exemplo: Determine as coordenadas polares $z = (\sqrt{3}, -1)$
$x \quad y$

$\rho^2 = (\sqrt{3})^2 + (-1)^2 \therefore \rho = 2$

$\operatorname{sen}\theta = \frac{-1}{2} \qquad \cos\theta = \frac{\sqrt{3}}{2}$



$\theta = 30° = \frac{\pi}{6}$

Como y é $(-)$ e $x = (+)$ está no 4º q

então $\theta = 2\pi - \frac{\pi}{6} = \frac{12\pi - \pi}{6} = \frac{11\pi}{6}$

Simplificação de potência de i

$i^n = i^r$, sendo r o $n \% 4$ $(r = n \% 4)$

Ex: Calcule i^{21}

$21 \% 4 = 1 \; ; \; i^{21} = i^1 = i$

$- i^{22}$

$22 \% 4 = 2 \; ; \; i^{22} = i^2 = -1$

$- i^{23}$

$23 \% 4 = 3 \; ; \; i^{23} = i^3 = -i$

$- i^4$

$4 \% 4 = 0 \qquad i^4 = i^0 = 1$

Nota: Conjugado de $z = (x, y) = x + yi$ é $x - yi$

- Forma algébrica $z = (x, y) = x + yi$
- Forma trigonométrica:

$\cos\theta = \frac{x}{\rho} \rightarrow x = \rho\cos\theta \qquad \operatorname{sen}\theta = \frac{y}{\rho} \rightarrow y = \rho\operatorname{sen}\theta$

$z = x + yi$

$z = \rho\cos\theta + \rho\operatorname{sen}\theta\, i \;\therefore\; z = \rho(\cos\theta + i\operatorname{sen}\theta)$

Operação na forma trigonométrica

$z_1 = \rho_1(\cos\theta_1 + i\operatorname{sen}\theta_1) \quad z_2 = \rho_2(\cos\theta_2 + i\operatorname{sen}\theta_2)$

$z_1 \cdot z_2 = \rho_1\rho_2(\cos(\theta_1 + \theta_2) + i\operatorname{sen}(\theta_1 + \theta_2))$

$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}(\cos(\theta_1 - \theta_2) + i\operatorname{sen}(\theta_1 - \theta_2))$

Identidade de Euler

$e^{i\theta} = \cos\theta + i\,\operatorname{sen}\theta$

$z = \rho(\cos\theta + i\,\operatorname{sen}\theta)$

$z = \rho e^{i\theta}$

Potenciação

Se $z=\rho(\cos\theta + i\,\operatorname{sen}\theta)$ então: $z^n = \rho^n(\cos(n\theta) + i\,\operatorname{sen}(n\theta))$

Ex: $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\,\operatorname{sen}\frac{\pi}{4}\right)$, determine z^4

$z^4 = (\sqrt{2})^4\left[\cos\left(4\cdot\frac{\pi}{4}\right) + i\,\operatorname{sen}\left(4\cdot\frac{\pi}{4}\right)\right]$

$= 4(\cos\pi + i\,\operatorname{sen}\pi)$

Algebricamente, temos $z^4 = 4(\cos\pi + i\,\cancel{\operatorname{sen}\pi}^{\,0})$ ∴ $z^4 = -4$

Radiciação

Se $z = \rho(\cos\theta + i\,\operatorname{sen}\theta) \rightarrow \sqrt[n]{z} = \sqrt[n]{\rho}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\,\operatorname{sen}\left(\frac{\theta+2\pi k}{n}\right)\right)$

sendo $k = 0, 1, 2, \dots, (n-1)$

Ex: $z = i$ achar $\sqrt[3]{z}$

1º) $z = (0,1)$ $\quad \rho^2 = 0^2 + 1^2$ ∴ $\rho = 1$

$\cos\theta = \frac{0}{1} = 0 \quad \operatorname{sen}\theta = \frac{1}{1} = 1$ ∴ $\theta = \frac{\pi}{2}$

$z = \rho(\cos\theta + i\,\operatorname{sen}\theta) = 1\left(\cos\frac{\pi}{2} + i\,\operatorname{sen}\frac{\pi}{2}\right)$

$\sqrt{z} = \cos\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right) + i\,\operatorname{sen}\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right)$

- Para $k = 0$

$\sqrt[3]{z} = \cos\left(\frac{\pi}{6}\right) + i\,\operatorname{sen}\left(\frac{\pi}{6}\right)$

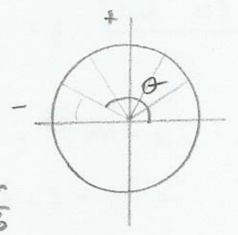
$= \frac{\sqrt{3}}{2} + i\frac{1}{2} = \frac{\sqrt{3} + i}{2}$

- Para $k = 1$

$\sqrt[3]{z} = \cos\left(\frac{\frac{\pi}{2} + 2\pi}{3}\right) + i\,\operatorname{sen}\left(\frac{\frac{\pi}{2} + 2\pi}{3}\right)$

$= \cos\left(\frac{5\pi}{6}\right) + i\,\operatorname{sen}\left(\frac{5\pi}{6}\right)$

$\frac{5\pi}{6}$ está no 2º q

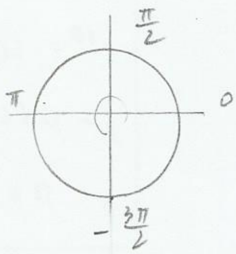


$\theta = \pi - \frac{5\pi}{6} = \frac{6\pi - 5\pi}{6} = \frac{\pi}{6}$

$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \quad \operatorname{sen}\frac{5\pi}{6} = \frac{1}{2}$

$$\sqrt[3]{z} = \cos\left(\frac{5\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{5\pi}{6}\right)$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i = \frac{-\sqrt{3}+i}{2}$$



Para $k=2$

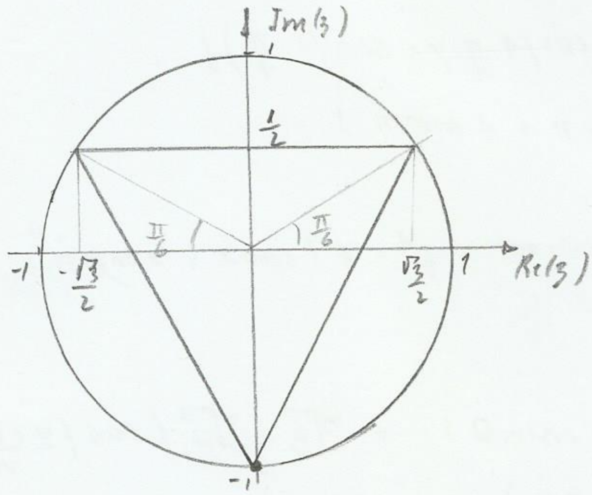
$$\sqrt[3]{z} = \cos\left(\frac{\frac{\pi}{2} + 2\pi \cdot 2}{3}\right) + i\,\mathrm{sen}\left(\frac{\frac{\pi}{2} + 2\pi \cdot 2}{3}\right)$$

$$= \cos\left(\frac{3\pi}{2}\right) + i\,\mathrm{sen}\left(\frac{3\pi}{2}\right)$$

$$= 0 - i$$

Representação no plano de Argand-Gauss

O raio → $r = \sqrt[n]{\rho}$



- O valor de n, refere-se ao número de vértices

Vértices: $\frac{\sqrt{3}+i}{2}$

$\frac{-\sqrt{3}+i}{2}$

$-i$

$\boxed{\text{Ex: 308}}$ $z = (+\sqrt{2} + 3i)(-\sqrt{2} + 3i)$

$$z = (3i + \sqrt{2})(3i - \sqrt{2}) = 9i^2 - 2 = -9 - 2 = -11$$

$$z = (x, y) = (-11, 0)$$

$$\rho = 11$$

$\boxed{\text{Ex: 309}}$ $z = \frac{25}{3+4i}$

$$z = \frac{25}{3+4i} \cdot \frac{(3-4i)}{(3-4i)} = \frac{75-100i}{9-16i^2} = \frac{75-100i}{25} = 3 - 4i$$

$$\rho = \sqrt{9+16} = 5$$

$\boxed{\text{Ex: 310}}$ $z = \frac{(i-\sqrt{3})^3}{4-4i}$

$$(i-\sqrt{3})^3 = (i-\sqrt{3})^2(i-\sqrt{3}) = [i^2 - 2i\sqrt{3} + (\sqrt{3})^2](i-\sqrt{3})$$

$$= (-1 - 2i\sqrt{3} + 3)(i - \sqrt{3}) = -i + \cancel{2\sqrt{3}} + 3i + \cancel{\sqrt{3}} + 6i - \cancel{3\sqrt{3}}$$

$$= 8i$$

$$z = \frac{8i}{4-4i} \cdot \frac{(4+4i)}{(4+4i)} = \frac{32i-32}{16+16} = i-1$$

$$\rho = \sqrt{1^2+1^2} = \sqrt{2}$$

$\boxed{Ex: 312}$ $z = \left(\frac{1+i}{1-i}\right)^4$

$$(1+i)^4 = (1+i)^2(1+i)^2 = (1+2i-1)(1+2i-1) = -4$$
$$(1-i)^4 = (1-i)^2(1-i)^2 = (1-2i-1)(1-2i-1) = -4$$

$$z = \frac{-4}{-4} = 1 \qquad \rho = 1$$

$\boxed{Ex: 314}$ $z = -2 = (-2,0)$

$$\rho = 2 \qquad \cos\theta = \frac{-2}{2} = -1 \quad \sin\theta = \frac{0}{2} = 0 \quad \therefore \alpha = \pi$$

$$z = 2(\cos\pi + i\sin\pi) = 2\,cis\,\pi$$

$\boxed{Ex: 315}$ $z = 3i = (0,3)$

$$\rho = 3 \qquad \cos\theta = \frac{0}{3} = 0 \quad \sin\theta = \frac{3}{3} = 1 \quad \therefore \alpha = \frac{\pi}{2}$$
$$z = 3\,cis\left(\frac{\pi}{2}\right)$$

Ex: 316 $z = 1+i = (1,1)$

$$z = \sqrt{2} \qquad \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \therefore \alpha = \frac{\pi}{4} \qquad z = \sqrt{2}\,cis\left(\frac{\pi}{4}\right)$$

Ex: 317 $z = -\sqrt{3}+i = (-\sqrt{3}, 1)$

$$\rho = \sqrt{(-\sqrt{3})^2+1^2} = 2 \qquad \cos\theta = \frac{-\sqrt{3}}{2} \quad \sin\theta = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6} \quad \therefore \quad \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \qquad \therefore \quad z = 2\,cis\left(\frac{5\pi}{6}\right)$$

318 $z = -\sqrt{2} - \sqrt{2}\,i = (-\sqrt{2}, -\sqrt{2})$

$$\rho = \sqrt{(-\sqrt{2})^2+(-\sqrt{2})^2} = 2 \qquad \cos\theta = \frac{-\sqrt{2}}{2} \quad \sin\theta = -\frac{\sqrt{2}}{2}$$

$$\alpha = \frac{\pi}{4} \qquad \theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \qquad \therefore \quad z = 2\,cis\left(\frac{5\pi}{4}\right)$$

319 $z = 1-i = (1,-1)$

$$\rho = \sqrt{2} \qquad \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \sin\theta = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\alpha = \frac{\pi}{4} \qquad \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4} \qquad \therefore \quad z = \sqrt{2}\,cis\left(\frac{7\pi}{4}\right)$$

320-) $z = 1 = (1,0)$

$\rho^2 = 1^2 + 0^2 \therefore \rho = 1 \qquad \cos\theta = \frac{1}{1} = 1 \quad \text{sen}\,\theta = \frac{0}{1} = 0$

$\theta = 0$ ou $2\pi \qquad \therefore z = \cos 0 + i\,\text{sen}\,0$

$\sqrt{z} = 1\left(\cos\left(\frac{0+2\pi k}{2}\right) + i\,\text{sen}\left(\frac{0+2\pi k}{2}\right)\right)$

$z^{\frac{1}{2}} = \cos(\pi k) + i\,\text{sen}(\pi k)$

- Para $k=0$

$z^{1/2} = \cos(\pi.0) + i\,\text{sen}(\pi.0) = 1$

$\boxed{R = \pm 1}$

- Para $k=1$

$z^{1/2} = \cos(\pi.1) + i\,\text{sen}(\pi.1) = -1$

321-) $z = 2i \Rightarrow z = (0,2)$

$\rho = \sqrt{0+4} = 2 \qquad \cos\theta = \frac{0}{2} = 0 \quad \text{sen}\,\theta = \frac{2}{2} = 1 \quad \therefore \theta = \frac{\pi}{2}$

$z = 2\left(\cos\frac{\pi}{2} + i\,\text{sen}\frac{\pi}{2}\right)$

$z^{\frac{1}{2}} = \sqrt{2}\left(\cos\left(\frac{\frac{\pi}{2}+2\pi k}{2}\right) + i\,\text{sen}\left(\frac{\frac{\pi}{2}+2\pi k}{2}\right)\right)$

Para $k=0$

$z^{1/2} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\,\text{sen}\left(\frac{\pi}{4}\right)\right)$

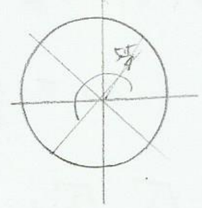
$= \sqrt{2}\cdot\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i$

Para $k=1$

$z^{1/2} = \sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\,\text{sen}\left(\frac{5\pi}{4}\right)\right)$

$= \sqrt{2}\left(-\cos\left(\frac{\pi}{4}\right) - i\,\text{sen}\left(\frac{\pi}{4}\right)\right)$

$= \sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -1 - i$



$\boxed{R = \pm(1+i)}$

322 $z = -i \rightarrow z = (0, -1)$

$\rho = 1 \quad \cos\theta = \frac{0}{1} = 0 \quad \sin\theta = \frac{-1}{1} = -1 \quad \therefore \theta = \frac{3\pi}{2}$

$z = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

$z^{1/2} = \cos\left(\frac{\frac{3\pi}{2} + 2\pi k}{2}\right) + i\sin\left(\frac{\frac{3\pi}{2} + 2\pi k}{2}\right)$

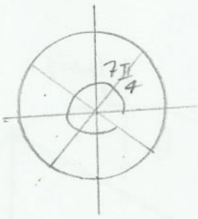
Para $k = 0$

$\pi - \frac{3\pi}{4} = \frac{\pi}{4}$

$z^{1/2} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

$= -\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$



$2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

Para $k = 1$

$z^{1/2} = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)$

$= \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)$

$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

$R = \pm\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)$

323: $z = 1 + \sqrt{3} i \rightarrow z = (1, \sqrt{3})$

$\rho = \sqrt{1^2 + (\sqrt{3})^2} = 2 \quad \cos\theta = \frac{1}{2} \quad \sin\theta = \frac{\sqrt{3}}{2} \quad \therefore \theta = \frac{\pi}{3}$

$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

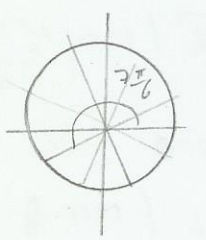
$z^{1/2} = \sqrt{2}\left(\cos\left(\frac{\frac{\pi}{3} + 2\pi k}{2}\right) + i\sin\left(\frac{\frac{\pi}{3} + 2\pi k}{2}\right)\right)$

Para $k = 0$

$z^{1/2} = \sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right) = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2} i$

Para $k = 1$



$z^{1/2} = \sqrt{2}\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$

$= \sqrt{2}\left(-\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$

$= \sqrt{2}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2} i$

$R = \pm\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2} i\right)$

352-1) $z^{\frac{1}{3}}$; $z=-1$ -> $z=(-1,0)$

$\rho = 1 \quad \cos\theta = \frac{-1}{1} = -1 \quad \sin\theta = \frac{0}{1} = 0 \quad \therefore \theta = \pi$

$z = \cos\pi + i\sin\pi$

$z^{1/3} = \sqrt[3]{z} = \cos\left(\frac{\pi + 2\pi . k}{3}\right) + i\sin\left(\frac{\pi + 2\pi k}{2}\right)$

Para $k=0$

$z^{1/3} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

$= \frac{1}{2} + \frac{\sqrt{3}}{2} i$

Para $k=1$

$z^{1/3} = \cos\pi + i\sin\pi$

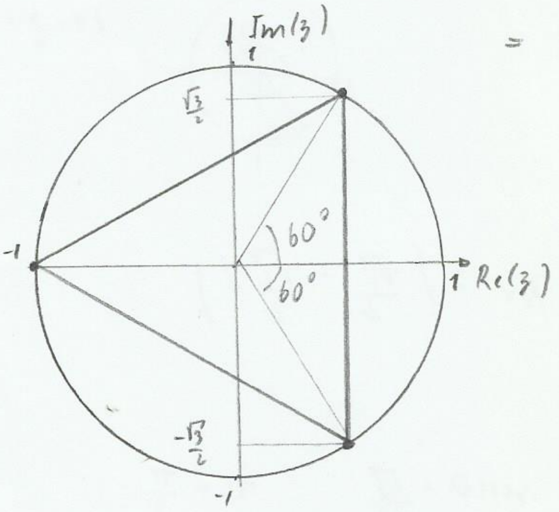
$= -1$

Para $k=2$

$z^{1/3} = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)$

$= -\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2} i$



354-1) $z^{1/4}$; $z = -8 + 8\sqrt{3} i$ -> $z = (-8, 8\sqrt{3})$

$\rho = \sqrt{8^2 + (8\sqrt{3})^2} = \sqrt{64 + 64\,3} = 16$

$\cos\theta = \frac{-8}{16} = \frac{-1}{2} \quad \sin\theta = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2} \quad \therefore \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$z = 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$z^{1/4} = \sqrt[4]{16}\left(\cos\left(\frac{\frac{2\pi}{3} + 2\pi k}{4}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi k}{4}\right)\right)$

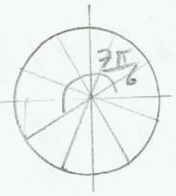
Para $k=0$

$z^{1/4} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$

Para $k=1$

$$z^{1/4} = 2\left(\cos\frac{2\pi}{3} + i\,\text{sen}\,\frac{2\pi}{3}\right)$$
$$= 2\left(-\cos\frac{\pi}{3} + i\,\text{sen}\,\frac{2\pi}{3}\right)$$
$$= 2\left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) = -1 + \sqrt{3}\,i$$



Para $k=2$

$$z^{1/4} = 2\left(\cos\frac{7\pi}{6} + i\,\text{sen}\,\frac{7\pi}{6}\right)$$
$$= 2\left(-\cos\frac{\pi}{6} + i\,\text{sen}\,\frac{\pi}{6}\right)$$
$$= 2\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) = -\sqrt{3} + i$$

Para $k=3$

$$z^{1/4} = 2\left(\cos\frac{5\pi}{3} + i\,\text{sen}\,\frac{5\pi}{3}\right)$$
$$= 2\left(\cos\frac{\pi}{3} - i\,\text{sen}\,\frac{\pi}{3}\right)$$
$$= 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - \sqrt{3}\,i$$

– Outra forma de calcular Raiz quadrada

+ calcule z^2 ; $z = 3+4i$

$$z = (x+yi) \to z^2 = x^2 + 2xyi - y^2 = 3+4i$$

$$\therefore \begin{cases} x^2 - y^2 = 3 \\ 2xy = 4 \end{cases}$$ Resolve o sistema e obtem

$w_1 = 2+i$

$w_2 = -2-i$

* Calcule módulo, parte real e a parte imaginária de $z = \dfrac{(\sqrt{3}+i)^6}{(\sqrt{2}+\sqrt{3}i)^6}$

$z_1 = \sqrt{3} + i \to z_1 = (\sqrt{3}, 1)$

$\rho = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \qquad \cos\theta = \dfrac{\sqrt{3}}{2} \quad \text{sen}\,\theta = \dfrac{1}{2} \quad \therefore \theta = \dfrac{\pi}{6}$

$z_1 = 2e^{\frac{\pi}{6}i}$

$z_2 = \sqrt{2} + \sqrt{3}i \to z_2 = (\sqrt{2}, \sqrt{3})$

$\rho = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2 \qquad \cos\theta = \dfrac{\sqrt{2}}{2} \quad \text{sen}\,\theta = \dfrac{\sqrt{2}}{2} \quad \therefore \theta = \dfrac{\pi}{4}$

$z_2 = 2e^{i\frac{\pi}{4}}$

$$z = \frac{(\sqrt{3}+i)^8}{(\sqrt{2}+\sqrt{2}i)^6} = \frac{(2e^{\frac{i\pi}{6}})^8}{(2e^{\frac{i\pi}{4}})^6} = \frac{2^8 e^{\frac{4\pi i}{3}}}{2^6 e^{\frac{i\pi 3}{2}}} = \frac{4e^{(4\pi i/3)}}{e^{(3\pi i/2)}}$$

$$= 4 \cdot e^{\left(\frac{4\pi i}{3} - \frac{3\pi i}{2}\right)}$$

$$= 4e^{\frac{-\pi}{6}i} \qquad \frac{4\pi i}{3} - \frac{3\pi i}{2} = \frac{8\pi i - 9\pi i}{6} = -\frac{\pi}{6}i$$

$$\therefore z = 4\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$$

$$= 4\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = 2\sqrt{3} - 2i$$

$$\therefore \rho = 4 \quad Im(z) = -2 \quad Re(z) = +2\sqrt{3}$$

Equações

Ex: 330) $(2+i)z - 8 + i = 0$

$$(2+i)z = 8 - i$$

$$z = \frac{8-i}{2+i} \cdot \frac{(2-i)}{(2-i)} = \frac{16 - 2i - 8i - 1}{4+1} = \frac{15 - 10i}{5} = \boxed{3 - 2i}$$

Ex: 331)

$$\frac{z+i}{\bar{z}+i} = \frac{-1+3i}{2}$$

$2(z+i) = (-1+3i)(\bar{z}+i)$; como $z = x + yi$ então $\bar{z} = x - yi$

$$2(x + yi + i) = (-1+3i)(x - yi + i)$$

$$2x + 2yi + 2i = -x + yi - i + 3xi + 3y - 3$$

$$\begin{cases} 2x + x - 3y + 3 = 0 \\ 2y + 2 - y + 1 - 3x = 0 \end{cases} \sim \begin{cases} 3x - 3y = -3 \\ -3x + y = -3 \end{cases} \sim \begin{cases} 3x - 3y = -3 \\ -2y = -6 \end{cases}$$

$$y = 3 \quad \therefore \quad x = 2 \quad \therefore \quad \boxed{z = 2 + 3i}$$

Ex: 333-)

$$z^2+2iz+3=0$$

$$z=\frac{-2i\pm\sqrt{(2i)^2-4.3.1}}{2}=\frac{-2i\pm\sqrt{-4-12}}{2}=\frac{-2i\pm 4i}{2}=-i\pm 4i$$

$$\boxed{\begin{array}{l} z_1=-5i \\ z_2=3i \end{array}}$$

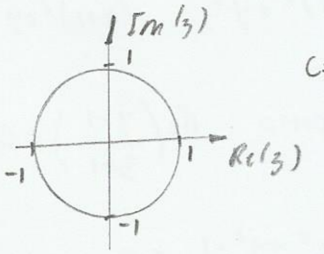
- Lugar geométrico

336-) $|z|=1$

$$z=x+yi=(x,y)$$

$$|z|=\sqrt{x^2+y^2}=1$$

$$x^2+y^2=1$$



$C=0,0$ e $r=1$

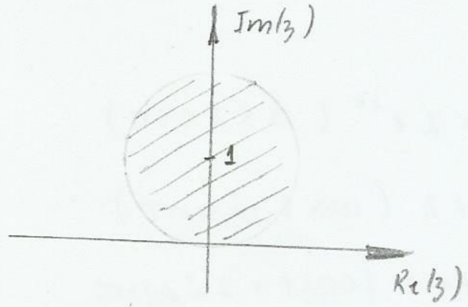
336-) $|z-i|\leq 1$

$$z=x+yi$$

$$|x+yi-i|\leq 1$$

$$|x+(y-1)i|\leq 1$$

$$x^2+(y-1)^2\leq 1$$



- (Prova antiga)

$|z-2|=|z-2i|$; $z=x+yi$

$$|x+yi-2|=|x+yi-2i|$$

$$|(x-2)+yi|=|x+(y-2)i|$$

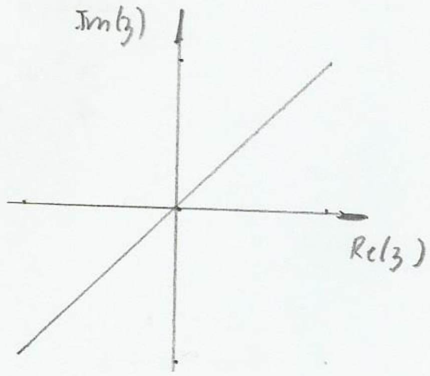
$$(x-2)^2+y^2=x^2+(y-2)^2$$

$$\cancel{x^2}-4x+4\cancel{+y^2}=\cancel{x^2}\cancel{+y^2}-4y+4$$

$$-4x+4=-4y+4$$

$$-x=-y$$

$$x=y$$



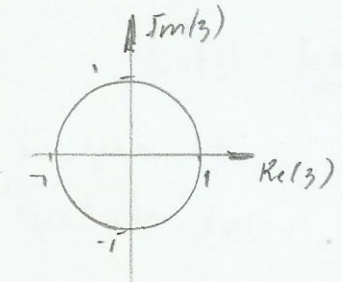
342) $Re\left(\frac{z-1}{z+1}\right)=0;\ z\neq -1$

$z = x+yi \qquad \frac{x+yi-1}{x+yi+1} = \frac{(x-1)+yi}{(x+1)+yi}\cdot\frac{(x+1)-yi}{(x+1)-yi}$

$$= \frac{(x-1)(x+1)+(x+1)yi-(x-1)yi-(yi)^2}{(x+1)^2-(yi)^2}$$

$$= \frac{x^2-1+\cancel{xyi}+yi-\cancel{xyi}+yi+y^2}{(x+1)^2+y^2} = \frac{x^2+y^2-1+2yi}{(x+1)^2+y^2}$$

$$= \frac{x^2+y^2-1}{(x+1)^2+y^2} + \frac{2yi}{(x+1)^2+y^2}$$



Como $Re\left(\frac{z-1}{z+1}\right)=0$

$\frac{x^2+y^2-1}{(x+1)^2+y^2}=0 \quad \therefore\ x^2+y^2-1=0$

$x^2+y^2=1$

341)

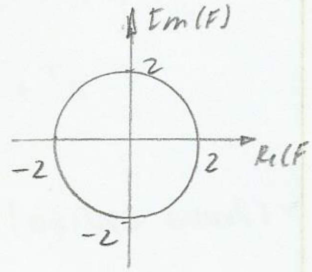
$z = 2e^{it}\ (0<t<2\pi)$

$x+yi = 2(\cos t + i\,\text{sen}\,t)$

$x+yi = 2\cos t + 2i\,\text{sen}\,t$

$x = 2\cos t \qquad \cos t = \frac{x}{2} \quad \text{sen}\,t = \frac{y}{2}$

$y = 2\,\text{sen}\,t$



$\cos^2 t + \text{sen}^2 t = 1$

$\frac{x^2}{4}+\frac{y^2}{4}=1 \quad \therefore\ x^2+y^2=4$

C=(0,0)

r=2

Mostrar que $f(z) = e^z$ transforma a reta $x=0$

$$e^z = e^{x+yi} = e^x e^{yi}$$
$$= e^x(\cos y + i \sin y)$$
$$= e^x \cos y + i\, e^x \sin y$$

$u = e^x \cos y$ Para $x=0$ temos $u = \cos y$ $v = \sin y$

$v = e^x \sin y$ $u^2 + v^2 = \cos^2 y + \sin^2 y = 1$

$*$ $f(z) = e^{2z}$

$$= e^{2(x+yi)} = e^{2x+2yi} = e^{2x} e^{2yi}$$
$$= e^{2x}(\cos 2y + i \sin 2y)$$
$$= \underbrace{e^{2x} \cos y}_{u} + i\, \underbrace{e^{2x} \sin 2y}_{v}$$

$\boxed{\text{logaritimo de um número complexo}}$

$- f(z) = \log z$; $z = \rho e^{i\theta}$

$$= \log(\rho e^{i\theta}) = \log \rho + \log e^{i\theta} = \boxed{\log \rho + i(\theta + 2k\pi)}$$

Calcule

(a) $\log(\sqrt{3} + i)$

$(x,y) = (\sqrt{3}, 1)$ $\rho = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\sin\theta = \frac{1}{2}$ $\therefore \theta = \frac{\pi}{6}$

$z = 2e^{\frac{\pi}{6} i}$

$$\log(2e^{\frac{\pi}{6} i}) = \log 2 + \log e^{\frac{\pi}{6} i} = \log 2 + i\left(\frac{\pi}{6} + 2K\pi\right)$$

b) $\log(-1)$

$(x,y) = (-1,0)$

$\rho = 1 \quad \operatorname{sen}\theta = \frac{0}{1} = 0 \quad \cos\theta = \frac{-1}{1} = -1 \quad \therefore \theta = \pi$

$z = e^{\pi i} \quad \log(e^{\pi i}) = i(\pi + 2\pi K) \; ; K \in \mathbb{Z}$

c) $\log(\sqrt{3} - i) \qquad (x,y) = (\sqrt{3}, -1)$

$\rho = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2 \qquad \cos\theta = \frac{\sqrt{3}}{2} \quad \operatorname{sen}\theta = \frac{-1}{2}$

$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \qquad ; \quad z = 2e^{\frac{11\pi}{6}i}$

$\log(2e^{\frac{11\pi}{6}i}) = \log 2 + \log(e^{\frac{11}{6}\pi i})$

$= \log 2 + i\left(\frac{11}{6}\pi + 2K\pi\right)$; sendo $K \in \mathbb{Z}$

d) $\log(-e) \qquad (x,y) = (-e, 0)$

$\rho = e \qquad \cos\theta = \frac{-e}{e} = -1 \quad \operatorname{sen}\theta = \frac{0}{e} = 0 \quad \therefore \theta = \pi$

$z = e.e^{\pi i}$

$\log(e.e^{\pi i}) = \log e + \log(e^{\pi i}) = 1 + i(\pi + 2K\pi), K \in \mathbb{Z}$

* Escreva os exercícios seguintes na forma algébrica, sendo

$a^z = e^{z \log a}$

<u>Ex 396</u>

$z = i^{1-i} \qquad a^z = e^{z \log a} \rightarrow i^{1-i} = e^{(1-i)\log i}$

$i \rightarrow (x,y) = (0,1) \quad \rho = 1 \quad \operatorname{sen}\theta = \frac{1}{1} = 1 \quad \cos\theta = \frac{0}{1} = 0 \quad \therefore \theta = \frac{\pi}{2}$

$i = e^{\pi/2 i} \quad \therefore \quad \log i = \log e^{\pi/2 i} = i\left(\frac{\pi}{2} + 2\pi K\right)$; $K \in \mathbb{Z}$ O valor principal é $\frac{\pi}{2}i$

$i^{1-i} = e^{(1-i)\frac{\pi}{2}i} = e^{\frac{\pi}{2}i + \frac{\pi}{2}} = e^{\frac{\pi}{2}} e^{\frac{\pi}{2}i} = e^{\frac{\pi}{2}}(\cos\frac{\pi}{2} + i \operatorname{sen}\frac{\pi}{2})$

$i^{1-i} = i e^{\pi/2}$

Derivação complexa

- A derivada de $f(z)$ apenas existe se $f(z)$ é analítica

Condição necessária para $f(x+yi) = u(x+yi) + v(x+yi)i$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

- Ex: Verifique se $f(z)=\bar{z}$ é analítica

$f(z)=\bar{z} = x - yi \quad ; \quad u(x,y) = x \qquad v(x,y) = -y$

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$

$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0$

$\therefore$ Como a condição de Cauchy-Rieman não é satisfeita, então $f(z)$ não é derivável

- Ex: Verifique se $f(z) = e^{-z}$ é analítica

$\cos(z) = \cos(-z)$

$\sin(-z) = -\sin(z)$

$$f(x+yi) = e^{-(x+yi)} = e^{-x-yi} = e^{-x}e^{-yi}$$
$$= e^{-x}(\cos(-y) + i\sin(-y))$$
$$= e^{-x}(\cos(y) - i\sin(y))$$
$$= e^{-x}\cos y - e^{-x}\sin(y)i$$

$u(x,y) = e^{-x}\cos y \qquad v(x,y) = -e^{-x}\sin(y)$

$\frac{\partial u}{\partial x} = -e^{-x}\cos y \Leftrightarrow \frac{\partial v}{\partial y} = -e^{-x}\cos(y)$

$\frac{\partial u}{\partial y} = -e^{-x}\sin y \Leftrightarrow -\frac{\partial v}{\partial x} = e^{-x}\sin(y)$

$\therefore$ Como as condições de Cauchy-Rieman são satisfeita, então f é analítica, portanto derivável

Funções harmônicas

(Mentes tão bem
que para verdade
... Até chego imaginar...
que amo de

$f(x+yi) = u(x,y) + i\,v(x,y)$

$\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0$ $\quad \frac{\partial v^2}{\partial x^2} + \frac{\partial v^2}{\partial y^2} = 0 \;\therefore\; f(z)$ é harmônica

* Exercício caderno

$v(x,y) = 2xy - y$ é harmônica

$\frac{\partial v}{\partial x} = 2y \quad \frac{\partial v^2}{\partial x^2} = 0 \qquad \frac{\partial v}{\partial y} = 2x - 1 \quad \frac{\partial v^2}{\partial y^2} = 0$

$\therefore \frac{\partial v^2}{\partial x^2} + \frac{\partial v^2}{\partial y^2} = 0 \;\therefore$ É harmônica

Determine $u(x,y)$ para $f(x+yi) = u(x,y) + i\,v(x,y)$ ser analítica

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ e $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = 2x - 1 \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$

$u = \int 2x - 1\, dx \qquad u = \int -2y\, dy = -y^2 + C$

$= x^2 - x + \phi(y)$

$\therefore u(x,y) = x^2 - x - y^2 + C$

$f(z) = u(x,y) + i\,v(x,y)$

$= \underbrace{x^2 - x - y^2}_{z^2} + C + 2xyi - yi$

$\boxed{= z^2 - z + C}$

Obs:

$z = x + yi$

$\bar{z} = x - yi$

$iz = xi - y = -y + xi$

$z^2 = x^2 + 2xyi - y^2$

$iz^2 = x^2 i - 2xy - y^2 i$

$-z = -x - yi$

* Verifique se $u(x,y) = e^{-x}\cos y$ é harmônica

$$\frac{\partial u}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial u^2}{\partial x^2} = e^{-x}\cos y$$

$$\frac{\partial u}{\partial y} = -e^{-x}\operatorname{sen} y \qquad \frac{\partial u^2}{\partial y^2} = -e^{-x}\cos y$$

$$\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = e^{-x}\cos y - e^{-x}\cos y = 0 \qquad \therefore \ u(x,y) \text{ é } \underline{\text{harmônica}}$$

Determine $v(x,y)$ para $f(x+yi) = u(x,y) + i\,v(x,y)$ ser analítica

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow \frac{\partial v}{\partial y} = -e^{-x}\cos y \qquad \partial v = -e^{-x}\cos y \, dy$$

$$v = -e^{-x}\int \cos y \, dy = -e^{-x}\operatorname{sen} y + \phi(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$-e^{-x}\operatorname{sen} y = -(+e^{-x}\operatorname{sen} y + \phi'(x))$$

$$-e^{-x}\operatorname{sen} y = -e^{-x}\operatorname{sen} y - \phi'(x) \qquad \therefore \phi'(x) = 0 \qquad \phi = C$$

$$v = -e^{-x}\operatorname{sen} y + C$$

$$f(z) = f(x+yi) = e^{-x}\cos y - e^{-x}\operatorname{sen} y \, i + C$$

$$= e^{-x}(\cos y - \operatorname{sen} y \, i) + C$$

$$= e^{-x}e^{yi} = e^{-x+yi} = e^{-z} + C$$

* Verifique se $v(x,y) = 2xy - y$ é harmônica

$$\frac{\partial v}{\partial x} = 2y \quad \frac{\partial v^2}{\partial x^2} = 0 \qquad \frac{\partial v}{\partial y} = 2x - 1 \quad \frac{\partial v^2}{\partial y^2} = 0$$

Como $\frac{\partial v^2}{\partial x^2} + \frac{\partial v^2}{\partial y^2} = 0 \rightarrow 0 + 0 = 0 \therefore$ é harmômia

- Determine $v(x,y)$ para $f(x+yi) = u(x,y) + i v(x,y)$ ser analítica

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow \frac{\partial u}{\partial x} = 2x - 1 \quad \partial u = (2x-1)\,\partial x$$

$$u = \int 2x - 1 \, dx = x^2 - x + \Phi(y)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow -2y = \Phi'(y) \therefore \Phi(y) = \int -2y \, dy = -y^2 + C$$

$$u = x^2 - x - y^2 + C$$

$$f(x+yi) = 2xy - y + x^2 - x - y^2 + C$$

$$= z^2 - z + C$$

$z = x + yi$

$\bar{z} = x - yi$

$z^2 = x + 2xyi - y^2$

$-z = -x - yi$

(Prova antiga)

$$z = Re\left(\frac{z}{z+1}\right)$$

$$\left(\frac{z}{z+1}\right) = \frac{x+yi}{x+yi+1} = \frac{x+yi}{(x+1)+yi} \cdot \frac{(x+1)-yi}{(x+1)-yi}$$

$$= \frac{x(x+1) - xyi + y(x+1)i + y^2}{(x+1)^2 + y^2} = \frac{x^2 + x - \cancel{xyi} + \cancel{yxi} + yi + y^2}{(x+1)^2 + y^2}$$

$$= \frac{x^2 + x + y^2}{(x+1)^2 + y^2} + \frac{yi}{(x+1)^2 + y^2}$$

$$\frac{x^2 + x + y^2}{(x+1)^2 + y^2} = 2 \qquad x^2 + x + y^2 = 2x^2 + 4x + 2 + 2y^2$$

$$x^2 + y^2 + 3x + 2 = 0$$

$(x-a)^2+(y-b)^2=r^2$

$x^2-2xa+a^2+y^2-2by+b^2=r^2$

$x^2+y^2+3x+2=0$

$-2xa=3x \quad \therefore -2a=3 \quad a=-\frac{3}{2}$

$2yb=0 \quad \therefore b=0$

$\frac{9}{4}-r^2=2 \quad r^2=\frac{9}{4}-2 \quad \therefore r=\sqrt{\frac{1}{4}}=\frac{1}{2}$

$(x+\frac{3}{2})^2+(y-0)^2=\frac{1}{4}$

b) Calcule $w=i^{3+2i}$

$i^{3+2i}=e^{(3+2i)\log i}$

$\log i=?$ $\quad i\to(0,1) \quad \rho=1 \quad \cos\theta=\frac{0}{1}=0 \quad \sin\theta=\frac{1}{1}=1$

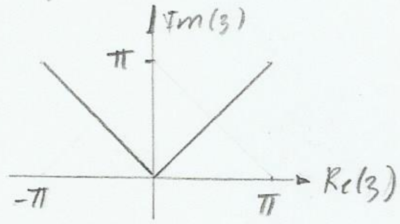
$\therefore \theta=\frac{\pi}{2} \qquad \log i=\log(e^{i\frac{\pi}{2}})=\frac{\pi}{2}i$

$i^{3+2i}=e^{(3+2i)\cdot\frac{\pi}{2}i}=e^{\frac{3\pi}{2}i-\pi}=e^{\frac{3\pi}{2}i}\,e^{-\pi}$

$=e^{-\pi}\left(\cancel{\cos\frac{3\pi}{2}}^{\,0}+i\sin\frac{3\pi}{2}\right)$

$=-e^{-\pi}\cdot i$

* Expanda a série de Fourier $f(x)=|-x|$



$f(x)=|-x| \begin{cases} x, & \text{se } x\geq 0 \\ -x, & \text{se } x<0 \end{cases}$

Par, pois é simétrico

$f(x)=f(-x) \quad \therefore b_n=0$

$a_0=\frac{2}{\pi}\int_0^\pi x\,dx=\frac{2}{\pi}\cdot\frac{x^2}{2}\Big|_0^\pi=\frac{1}{\pi}(\pi^2-0^2)=\pi$

$a_n=\frac{2}{\pi}\int_0^\pi x\cos(nx)dx$; $\quad x=u \quad dv=\cos(nx)$

$dx=du \quad v=\frac{1}{n}\sin(nx)$

$\int_0^\pi x\cos(nx)dx=\frac{x}{n}\sin(nx)+\frac{1}{n^2}\cos(nx)\Big|_0^\pi$

$=\frac{\pi}{n}\cancel{\sin(n\pi)}^{\,0}+\frac{1}{n^2}\cos(n\pi)-\frac{0}{n}\cancel{\sin(n0)}^{\,0}+\frac{1}{n^2}\cos^{1}(0.n)$

$a_n = \frac{1}{n^2}(\cos(n\pi)+1)$

$a_n = \frac{2}{\pi}\frac{1}{n^2}(\cos(n\pi)+1)$

$= \begin{cases} \frac{4}{\pi n^2} & \text{, se } n \text{ par} \\ 0 & \text{, se } n \text{ impar} \end{cases}$

* Escrever $z = \dfrac{(1+i)^{40} + (1-i)^{42}}{i^{46}}$ na forma algébrica

$(1+i) = (1,1)$

$\rho = \sqrt{2}$ $\quad \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad \sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad (1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$(1+i)^{40} = 2^{\frac{1}{2}\cdot 40}\left(\cos\left(\frac{40\pi}{4}\right) + i\sin\left(\frac{40\pi}{4}\right)\right) = 2^{20}(\cos 10\pi + i\sin 10\pi) = 2^{20}$

$(1-i) = (1,-1)$

$\rho = \sqrt{2}$ $\quad \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad \sin\theta = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$ $\quad \therefore \theta = \frac{\pi}{4}$

$1-i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$ $\qquad + \; - \quad 4°Q$

$\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$(1-i)^{42} = 2^{\frac{1}{2}\cdot 42}\left(\cos\left(\frac{42\cdot 7\pi}{4}\right) + i\sin\left(\frac{42\cdot 7\pi}{4}\right)\right)$

$= 2^{21}(-1)i = -i2^{21}$

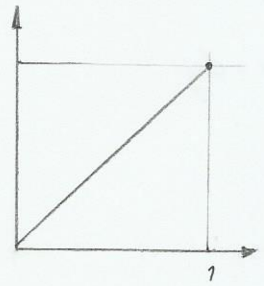
$46 \,\%\, 4 = 0$

$i^{46} = i^0 = 1$

$$\boxed{z = \frac{2^{20} - 2^{21}i}{1}}$$

Integral de linha

- Calcule $\int_C \bar{z}\,dz$, onde C é o segmento da reta que une $z_1(0,0)$ a $z_2=(1,1)$



$y = x$

$\therefore dy = dx$

$z = x + yi$

$dz = dx + dy\,i$

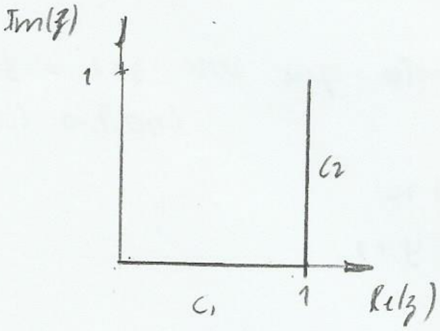
$$\int_C \bar{z}\,dz = \int_C (x-yi)\,dz = \int_C (x-yi)(dx+dy\,i)$$

$$= \int_0^1 x\,dx - \cancel{yi\,dx} + \cancel{x\,dy\,i} + y\,dy$$

$$= \frac{x^2}{2}\Big|_0^1 + \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}+\frac{1}{2} = \underline{1}$$

- $\int_C \bar{z}\,dz$, onde C é a poligonal que une $z=0$ $z=1$ $z=1+i$

$z_1=(x,y)=(0,0)$ $z_2=(1,0)$ $z_3=(1,1)$



$C_1: y=0 \quad dy=0$

$C_2: x=1 \quad dx=0$

$\int \bar{z}\,dz$

a) Em C_1

$$\begin{cases} y=0 \\ dy=0 \end{cases}$$

$$I_1 = \int_{C_1} (x-\overset{\to 0}{yi})(dx+\overset{\to 0}{dy\,i})$$

$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

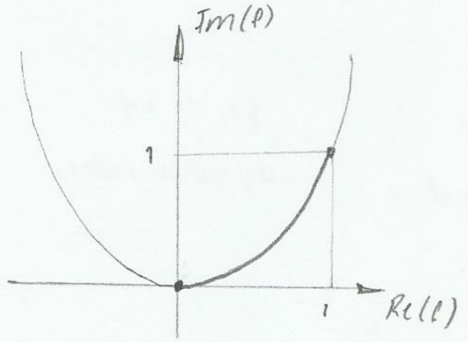
b) Em C_2

$$\begin{cases} x=1 \\ dx=0 \end{cases}$$

$$I_2 = \int_{C_2} (\overset{1}{x}-yi)(\overset{\to 0}{dx}+dy\,i) \Rightarrow \int (1-yi)\,dy\,i = \int_0^1 i\,dy + \int_0^1 y\,dy = iy + \frac{y^2}{2} = i + \frac{1}{2}$$

$\int_C \bar{z}\,dz = I_1 + I_2 = \frac{1}{2} + i + \frac{1}{2} = \underline{1+i}$

* $\int_C \bar{z}\,dz$, sendo $C \to y = x^2$ que une $z=0 \to (0,0)$ e $z=(1+i) \to (1,1)$



$x^2 = y$

$2x\,dx = dy$

$\int_C \bar{z}\,dz = \int_C (x - yi)\,dz$

$= \int_C (x-yi)(dx + i\,dy)$

$= \int_0^1 (x - x^2 i)(dx + i\,2x\,dx)$

$= \int_0^1 x\,dx - x^2 i\,dx + 2x^2 i\,dx + 2x \cdot x^2\,dx$

$= \int_0^1 x\,dx + x^2 i\,dx + 2x^3\,dx$

$= \frac{x^2}{2} + \frac{x^3}{3} i + \frac{2x^4}{4} \Big|_0^1 = \frac{1}{2} + \frac{1}{3} i + \frac{1}{2} = 1 + \frac{1}{3} i$

* $\int_C (z+i)\,dz$, sendo C o segmento de reta que une $z=1$ a $z=i$
$(1,0) \to (0,1)$

$f(z) = z + i$ $\qquad u = x$

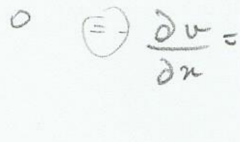
$= x + yi + i = x + (y+1)i$ $\qquad v = y+1$

$\frac{\partial u}{\partial x} = 1 \Leftrightarrow \frac{\partial v}{\partial y} = 1$

$\therefore f$ é analítica

$\frac{\partial u}{\partial y} = 0 \Leftrightarrow \frac{\partial v}{\partial x} = 0$

1).



$y = -x + 1$

$dy = -dx$

$$I = \int_0^1 x + (y+1)i \; (dx + dy\,i)$$

$$= \int_0^1 (x + yi + i)\,(dx + dy\,i)$$

$$= \int_0^1 (x - xi + i + i)(dx - dx\,i)$$

$$= \int_0^1 (x - xi + 2i)\,(dx - dx\,i)$$

$$= \int_0^1 (x\,dx - xi\,dx + 2i\,dx - xi\,dx - x\,dx + 2dx)$$

$$= \cancel{\frac{x^2}{2}} - \frac{x^2}{2}i + 2ix - \frac{x^2}{2}i \cancel{- \frac{x^2}{2}} + 2x \Big|_0^1$$

$$= -\frac{1}{2}i + 2i - \frac{1}{2}i + 2 = 2 + i = -2 - i$$

Exercícios de prova

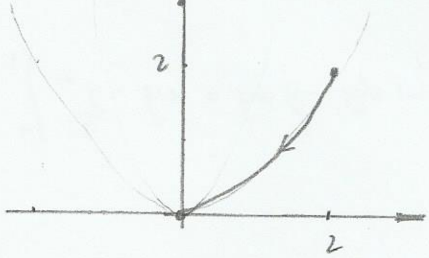
1-) $\int_C (\bar{z} + 2)\, dz$ C: $y = \frac{x^2}{2}$ que une $(2,2) \to (0,0)$

$f(z) = \bar{z} + 2 = x - yi + 2$ $u = x + 2$

$= (x+2) - yi$ $v = -y$

$\frac{\partial u}{\partial x} = 1$ $\frac{\partial v}{\partial y} = -1$ $\therefore$ f não é analítica

$\frac{\partial u}{\partial y} = 0$ $\frac{\partial v}{\partial x} = 1$



$y = \frac{x^2}{2}$

$dy = x\,dx$

$$\int_2^0 [(x+2) - yi].(dx + dy\,i)$$

$$\int_2^0 (x + 2 - \frac{x^2}{2}i)\,(dx + x\,dx\,i)$$

$$\int_2^0 x\,dx + 2dx - \frac{x^2}{2}i\,dx + x^2 i\,dx + 2xi\,dx + \frac{x^3}{2}dx$$

$$= \int_2^0 x\,dx + 2dx - \frac{x^2}{2}i\,dx + 2xi\,dx + \frac{x^3}{2}dx$$

$$= \frac{x^2}{2} + 2x + \frac{x^3}{6}i + x^2 i + \frac{x^4}{8}\Big|_2^0 = -\left(\frac{2^2}{2} + 2.2 + \frac{2^3}{6}i + 4i + \frac{2^4}{8}\right)$$

$\frac{8}{6} = \frac{4}{3}$ $\frac{16}{8} = 2$

2 4

$= -(8$

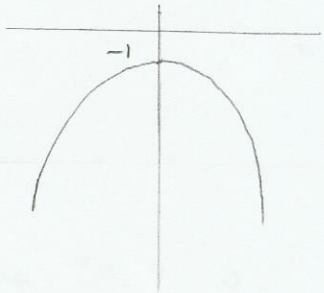
1-) $\int_c (\bar{z} - i)\, dz$

$y = -x^2 - 1$ unido $A = (0,1)$ até $B = (1,0)$

$f(z) = \bar{z} - i$ $\quad u = x$

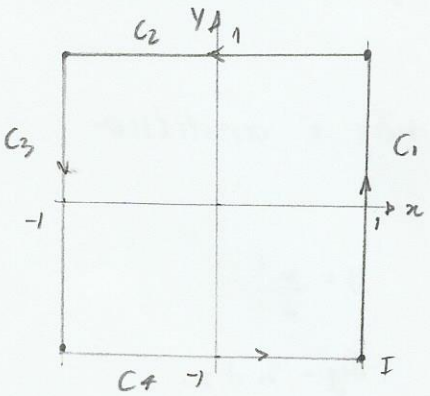
$= x - yi - i = x - (y+1)i$ $\quad v = -y - i$

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = -1 \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow 0 = 0$



X

1-) $\int \bar{z}\, dz$



$C_1: \; x = 1 \quad dx = 0$

$C_2: \; y = 1 \quad dy = 0$

$C_3: \; x = -1 \quad dx = 0$

$C_4: \; y = -1 \quad dy = 0$

$I_1 = \int (\overset{1}{x} + yi)(\overset{\to 0}{dx} + dy\, i) \rightarrow \int_{-1}^{1} (1 + yi)\, dy\, i \rightarrow \int_{-1}^{1} i\, dy - y\, dy = iy - \frac{y^2}{2} \Big|_{-1}^{1}$

$i - \frac{1}{2} - \left(-i - \frac{1}{2}\right) = 2i$

$I_2 = \int (x + \overset{1}{y} i)(dx + \overset{\to 0}{dy\, i}) \rightarrow \int_1^{-1} (x - i)\, dx = \int_1^{-1} x\, dx - i\, dx = \frac{x^2}{2} - ix \Big|_1^{-1}$

$= \frac{1}{2} + i - \left(\frac{1}{2} - i\right) = 2i$

$I_3 = \int (\overset{-1}{x} + yi)(\overset{\to 0}{dx} + dy\, i) \rightarrow \int_1^{-1} (-1 + yi)\, dy\, i \rightarrow \int_1^{-1} -i\, dy - y\, dy = -iy - \frac{y^2}{2} \Big|_1^{-1}$

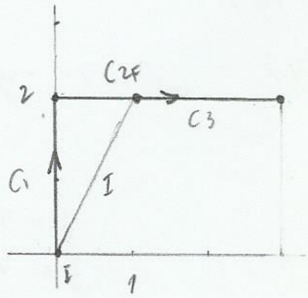
$= i - \frac{1}{2} - \left(-i - \frac{1}{2}\right) = 2i \qquad , \quad I_4 = 2i$

$\therefore \; I = 6i$

7.) $f(x+iy) = 5x - y + ix + 5iy$ $\qquad u = 5x - y$

$= (5x-y) + (x+5y)i \qquad v = x + 5y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \to 5 = 5 \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \to -1 = -(1) \quad \therefore$ é analítica



$C: \; y = 2x \quad dy = 2dx$

$I = \int_0^1 (5x - y + ix + 5iy)(dx + dy\,i)$

$= \int_0^1 (5x - 2x + ix + 10xi)(dx + 2dx\,i)$

$= \int_0^1 (3x\,dx + 11xi\,dx + 6x\,dx\,i - 22x\,dx)$

$= \int_0^1 -19x\,dx + 17xi\,dx = -\frac{19x^2}{2} + \frac{17x^2}{2} i \Big|_0^1 = -\frac{19}{2} + \frac{17}{2} i$

En Entra

$u(x,y) = 3x^2 - 3y^2 + 6x$ Determine $v(x,y)$ para $f(x+yi) = u(x,y) + iv(x,y)$

$\frac{\partial u}{\partial x} = 6x + 6 = \frac{\partial v}{\partial y} \quad \therefore \partial v = (6x+6)\,dy$

$v = \int 6x + 6\,dy = 6xy + 6y + \phi x$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -(6y + \phi' x) = -6y \qquad \phi'(x) = 0$

$6y + \phi' x = 6y \qquad \therefore \phi(x) = C$

$v = 6xy + 6y + C$

$f(z) = 3x^2 - 3y^2 + 6x + 6xyi + 6i + C$

$= 3(x^2 - y^2 + 2x + 2xyi + 2i) + C$

$= 3(z^2 + 2z) + C = 3z^2 + 6z + C$

* Se $v(x,y) = 5x^2 - 5y^2 + x$, determine $u(x,y)$, para $f(x+yi) = u(x,y) + iv(x,y)$ para ser analítica

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow -10y = \frac{\partial u}{\partial x} \quad u = \int -10y \, dx = -10yx + \phi(y)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow -(10x+1) = -10x + \phi'(y)$$

$$\cancel{-10x} - 1 = \cancel{-10x} + \phi'(y) \quad \therefore \phi(y) = -y + C$$

$$u = -10yx - y + C$$

$$f(z) = (5x^2 - 5y^2 + x)i - 10yx - y + C$$

$$= 5x^2 i - 5y^2 i + xi - 10yx - y + C$$

$$= 5(x^2 i - y^2 i - 2yx) + iz$$

$$\boxed{= 5iz^2 + iz + C}$$

$z = x + yi$

$\bar{z} = x - yi$

$z^2 = x^2 + 2xyi - y^2$

$-z^2 = -x^2 - 2xyi - y^2$

$iz^2 = ix^2 - 2xy - iy^2$

$-iz^2 = -ix^2 + 2xy + iy^2$

$iz = xi - y$

Limites de funções complexas

a) $\lim\limits_{z \to i} \dfrac{z^2 - z + 1 + i}{z^2 - 2z + 1}$

$$\frac{i^2 - i + 1 + i}{i^2 - 2i + 1} = \frac{-1 - i + 1 + i}{-1 - 2i + 1} = \frac{0}{-2i} = 0$$

b) $\lim\limits_{z \to 3i} \dfrac{z^2 - 2iz + 3}{z^2 + (1-3i)z - 3i}$

$$\frac{-9 + 6 + 3}{-9 + (1-3i).3i - 3i} = \frac{-9+9}{-9 + 3i + 9 - 3i} = \frac{0}{0}$$

$$\overset{LH}{=} \lim_{z \to 3i} \frac{2z - 2i}{2z + 1 - 3i} = \frac{6i - 2i}{6i + 1 - 3i} = \frac{4i}{3i+1} \cdot \frac{(-3i+1)}{(-3i+1)} = \frac{12 + 4i}{1 + 9}$$

$$= \frac{12 + 4i}{10} = \frac{6 + 2i}{5}$$

Teorema de Cauchy

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Exemplos

1) $\oint \frac{z^3}{(z-1)^3}$ onde $C: |z|=4$

$f(z)=z^3$

$a=1$

$n+1=3 \therefore n=2 \to f''(z)=6z$

$f(z)=z^3$
$f'(z)=3z^2$
$f''(z)=6z$

$$\oint \frac{z^3}{(z-1)^3} = \frac{2\pi i}{2!}\,6.1 = 6\pi i$$

2) $\oint_C \frac{z}{z+i}\,dz$

$f(z)=z$

$n+1=1 \therefore n=0$ (Não tem derivada)

$a=-i$

$$\oint_C \frac{z}{z+i}\,dz = \frac{2\pi i}{0!}\,-i = 2\pi$$

3) $\oint_C \frac{\cos z}{z-5}\,dz \quad C: |z|=2$

Como a é externo a C $\therefore$ $\oint_C \frac{\cos z}{z-5}\,dz = 0$

4) $\oint \frac{z^2-2z+1}{z^2(z-3)}\,dz \quad C\,|z|=2$

$$\oint \frac{(z^2-2z+1)(z-3)^{-1}}{z^2}\,dz$$

$$f(z)=\frac{z^2-2z+1}{z-3}$$

$a=0$

$n+1=2 \to n=1 \therefore f'(z)=?$

$$f'(z)=\frac{(2z-2)(z-3)+(z^2-2z+1)}{(z-3)^2}$$

$$=\frac{2z^2-2z-6z+6+z^2+2z-1}{(z-3)^2}$$

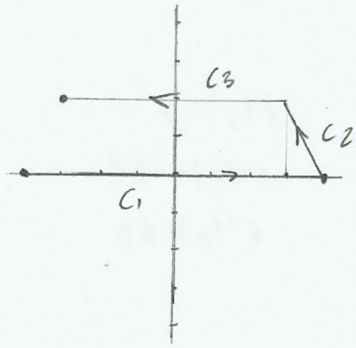
$$=\frac{3z^2-10z+5}{(z-3)^2}$$

$$f'(0)=\frac{5}{9}$$

$$= \frac{2\pi i}{1} \cdot \frac{5}{9} = \frac{10\pi i}{9}$$

(Prova antigo)

Calcule $\int (\bar{z}+1)\,dz$ onde vai $(-4,0) \to (4,0) \to (3,2) \to (-3,2)$



$f(z) = \bar{z}+1$ $\qquad u = x+1$

$= x - yi + 1$ $\qquad v = -y$

$= x+1-yi$.

(PULEI)

(Prova antiga)

$v(x,y) = 5x^2 - 5y^2 + x$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \to \frac{\partial u}{\partial x} = -10y \qquad du = -10y\,dx$

$u = -10yx + \phi y$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \to -10x + \phi' y = -(10x+1)$

$\cancel{-10x} + \phi' y = \cancel{-10x} - 1 \quad \therefore \phi(y) = -y + C$

$u = -10yx - y + C$

$f(z) = (5x^2 - 5y^2 + x)i - 10yx - y$

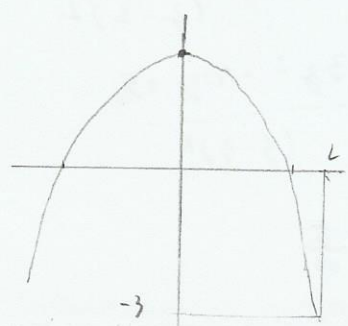
$= 5x^2 i - 5y^2 i + xi - 10yx - y$

$= 5(x^2 i - y^2 i - 2yx) + xi - y$

$= 5iz^2 + iz + C$

2 (Prova antiga)

$\int_C (z+i)\,dz \quad C: \; y = -x^2+1 \quad A = (0,1)$ a $B: (2,-3)$



$y = -x^2 + 1$

$dy = -2x\,dx$

$\int_C (z+i)\,dz$

$u = x$

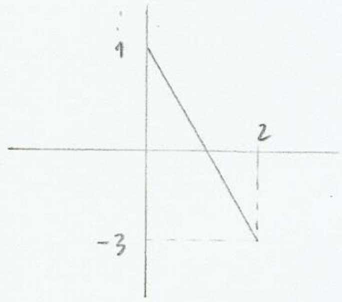
$v = y+1$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \to 1 = 1$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \to 0 = 0$

$f(z) = z+i$

$= x + yi + i = x + (y+1)i$



$y = -2x+1$

$dy = -2\,dx$

$\int_0^2 (x + (y+1)i)(dx + dy\,i) \to \int_0^2 (x - 2xi + i)(dx - 2dx\,i)$

$\to \int_0^2 x\,dx - 2xi\,dx + i\,dx - 2xi\,dx - 4x\,dx + 2\,dx$

$\to \int_0^2 -3x\,dx - 4xi\,dx + i\,dx + 2\,dx$

$= -\frac{3x^2}{2} - 2x^2 i + ix + 2x \Big|_0^2 = -6 - 8i + 2 + 4 = -8i$

$* \oint_c \frac{z^2+2z-1}{(z-i)^2(z-2)} dz \quad c\,|z|=3$

$$I = \oint_{c_1} \frac{(z^2+2z-1)(z-2)^{-1}}{(z-i)^2} dz + \oint_{c_2} \frac{(z^2+2z-1)(z-i)^{-2}}{(z-2)}$$

I_1 $f(z) = \frac{z^2+2z-1}{z-2}$ $\quad f'(z) = \frac{(2z+2)(z-2)-(z^2+2z-1)}{(z-2)^2}$

$a = i$ $\quad = \frac{(2i+2)(i-2)-(i^2+2i-1)}{(i-2)^2}$

$n+1=2 \Rightarrow n=1$

$$= \frac{-\cancel{2}+\cancel{2i}-4i-4+\cancel{1}-\cancel{2i}+\cancel{1}}{i^2-4i+4}$$

$$= \frac{-4i-4}{-4i+3} \cdot \frac{(3+4i)}{(3+4i)} = \frac{-12i-12+16-16i}{9+16} = \frac{-28i+4}{25}$$

I_2 $\quad I_1 = \frac{2\pi i}{1} \cdot \left(\frac{-28i+4}{25}\right)$

$f(z) = \frac{z^2+2z-1}{(z-i)^2}$

$a=2$ $\quad f(2) = \frac{2^2+2.2-1}{(2-i)^2} = \frac{7}{4-4i-1}$

$n+1=1 \Rightarrow n=0$

$$= \frac{7}{3-4i} \cdot \frac{(3+4i)}{(3+4i)} = \frac{21+28i}{9+16} = \frac{21+28i}{25}$$

$$I_2 = \frac{2\pi i}{1} \cdot \left(\frac{21+28i}{25}\right)$$

$$I = \frac{\cancel{56\pi}+8\pi i+42\pi i-\cancel{56\pi}}{25} = \underline{2\pi i}$$

(1) $z = \dfrac{(\sqrt{3}+i)^8}{(\sqrt{2}+\sqrt{2}i)^6}$

$z_1 = \sqrt{3} + i \to (\sqrt{3}, 1)$ $\qquad \cos\theta = \frac{\sqrt{3}}{2} \quad \sin\theta = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{6}$

$\rho = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$z_1 = 2e^{i\pi/6}$

$z_2 = \sqrt{2} + \sqrt{2}i \to (\sqrt{2}, \sqrt{2})$

$\rho = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$ $\qquad \cos\theta = \frac{\sqrt{2}}{2} \quad \sin\theta = \frac{\sqrt{2}}{2} \quad \therefore \theta = \frac{\pi}{4}$

$z_2 = 2e^{\frac{i\pi}{4}}$

$z = \dfrac{(2e^{i\pi/6})^8}{(2e^{i\frac{\pi}{4}})^6} = \dfrac{2^{\cancel{8}\,2} \cdot e^{i\pi\frac{4}{3}}}{\cancel{2^6}\, e^{i\frac{\pi}{4}6}} = \dfrac{4 \cdot (\cos\frac{4\pi}{3} + \sin\frac{4\pi}{3})}{\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}}$

$= \dfrac{4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)}{-i} = \dfrac{-2 - 2\sqrt{3}i}{-i} \cdot \dfrac{i}{i} = \dfrac{2\sqrt{3} - 2i}{1}$

$v(x,y) = 5x^2 - 5y^2 + x$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ e $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\dfrac{\partial v}{\partial y} = -10y = \dfrac{du}{dx} \to du = -10y\,dx$

$u = -10yx + \emptyset y$

$\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x} \to \cancel{-10x} - 1 = \cancel{-10x} + \emptyset' y \quad \therefore \emptyset(y) = -y + C$

$f(z) = (5x^2 - 5y^2 + x)i - 10yx - y + C$

$= 5x^2 i - 5y^2 i + xi - 10yx - y + C$

$= 5(x^2 i - y^2 i - 2yx) + xi - y + C$

$= \underline{5iz^2 + zi + C}$

$z = x + yi$

$\bar{z} = x - yi$

$z^2 = x^2 + 2xyi - y^2$

$-z^2 = -x^2 - 2xyi + y^2$

$iz^2 = x^2 i - 2xy - y^2 i$

$zi = xi - y$

Para $k=1$

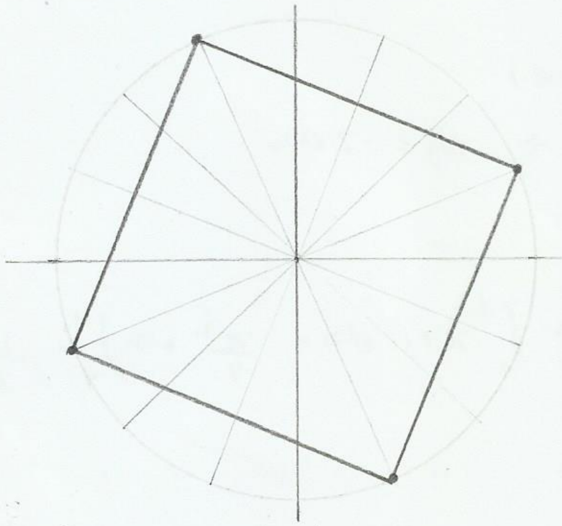
$$w_2 = 2\left(\cos\frac{5\pi}{8} + i\,\mathrm{sen}\,\frac{5\pi}{8}\right)$$

Para $k=2$

$$w_3 = 2\left(\cos\frac{9\pi}{8} + i\,\mathrm{sen}\,\frac{9\pi}{8}\right)$$

Para $k=3$

$$w_4 = 2\left(\cos\frac{13\pi}{8} + i\,\mathrm{sen}\,\frac{13\pi}{8}\right)$$



(4) $|z-2| = |z-2i|$

$z = x + yi$

$|x+yi-2| = |x+yi-2i|$

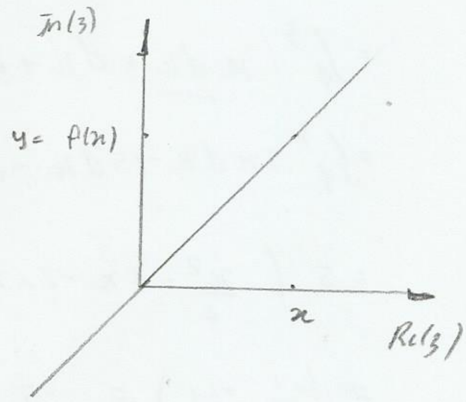
$|(x-2)+yi| = |x+(y-2)i|$

$(x-2)^2 + y^2 = x^2 + (y-2)^2$

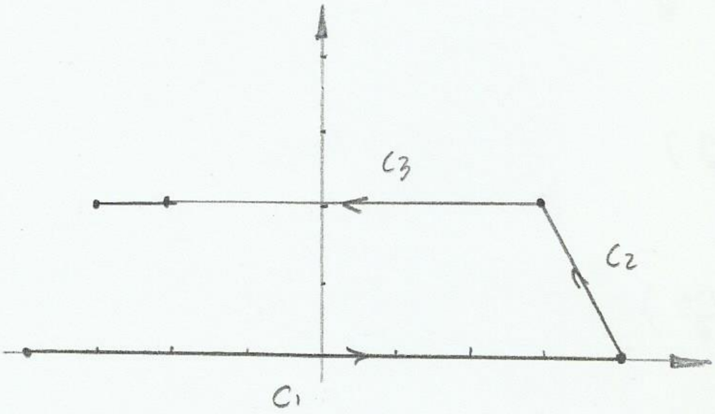
$\cancel{x^2} - 4x \cancel{+4} \cancel{+y^2} = \cancel{x^2} \cancel{+y^2} - 4y \cancel{+4}$

$-x = -y$

$x = y$



⑤ $\int_C (\bar{z}+1)\,dz$, C: $(-4,0) \rightarrow (4,0) \rightarrow (3,2) \rightarrow (-3,2)$



$f(z) = \bar{z} + 1$
$= x - yi + 1$
$= x + 1 - yi$

$u = x + 1$
$v = -y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow 1 = -1$

$\therefore$ f não é analítico

C_1: $y = 0$ $dy = 0$

C_2: $(y-0) = -2(x-4)$
$y = -2x + 8 \rightarrow dy = -2dx$

C_3: $y = 2$ $dy = 0$

$I_1 = \int_{C_1} (x+1-\cancel{yi}^{\,0})(dx+\cancel{i dy}^{\,0}) \rightarrow \int_{-4}^{4} x+1\,dx = \frac{x^2}{2} + x \Big|_{-4}^{4} = \frac{16}{2} + 4 - \frac{16}{2} + 4 = 8$

$I_2 = \int_{C_2} [(x+1) - yi].(dx + i dy)$

$= \int_4^3 (x+1+2xi-8i)(dx - 2i dx)$

$= \int_4^3 x\,dx + dx + 2xi\,dx - 8i\,dx - 2xi\,dx - 2i\,dx + 4x\,dx - 16\,dx$

$= \int_4^3 5x\,dx - 15\,dx - 10i\,dx = 5(\int_4^3 x\,dx - 3\,dx - 2i\,dx)$

$= 5\left(\frac{x^2}{2} - 3x - 2ix \Big|_4^3\right) = 5.\left(\frac{9}{2} - 9 - 6i - \frac{16}{2} + 12 + 8i\right)$

$= 5\left(\frac{-1}{2} + 2i\right) = \frac{-5}{2} + 10i$

$I_3 = \int_{C_3} [(x+1) - y^2 i](dx + \cancel{dy i}^{\,0})$

$\int_3^{-3} (x+1-2i)\,dx = \int_3^{-3} x\,dx + dx - 2i\,dx = \frac{x^2}{2} + x - 2ix \Big|_3^{-3}$

$= \cancel{\frac{9}{2}} - 3 + 6i - \cancel{\frac{9}{2}} - 3 + 6i = \boxed{12i - 6}$

$*\quad (2-2i)^{1+i}$

$+\quad (2-2i)(2-2i)^{i}$

$$(2-2i)^{i} = e^{i\log(2-2i)}$$

$\log(2-2i) \qquad f(z) = (2-2i) \to (2,-2)$

$$\rho = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2} \qquad \cos\theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \sin\theta = \frac{-2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\therefore \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$$f(z) = 2\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 2\sqrt{2}\, e^{i\frac{7\pi}{4}}$$

$$\log(2-2i) = \log(2\sqrt{2}\cdot e^{i\frac{7\pi}{4}}) = \log 2\sqrt{2} + i\left(\frac{7\pi}{4} + 2\pi k\right) \; ; \; k=0$$

$$= \log 2\sqrt{2} + \frac{7\pi i}{4}$$

$$\therefore (2-2i)^{i} = e^{i\left(\log 2\sqrt{2} + \frac{7\pi i}{4}\right)} \quad \ldots$$

OU

$$(2-2i)^{1+i} = e^{(1+i)\log(2-2i)}$$

$$(1+i)\left(\log 2\sqrt{2} + \frac{7\pi i}{4}\right)$$

$$\log(2-2i) = \log 2\sqrt{2} + \frac{7\pi i}{4}$$

$$\log 2\sqrt{2} + \frac{7\pi i}{4} + i\log 2\sqrt{2} - \frac{7\pi}{4}$$

$$\log 2\sqrt{2} - \frac{7\pi}{4} + \left(\log 2\sqrt{2} + \frac{7\pi}{4}\right)i$$

$$= e^{\log 2\sqrt{2} - \frac{7\pi}{4}} \cdot e\left(\cos\left(\log 2\sqrt{2} + i\frac{7\pi}{4}\right) + i\sin\left(\log 2\sqrt{2} + \frac{7\pi}{4}\right)\right)$$

Dada $v(x,y) = 2x - 2xy$ Determine $u(x,y)$ para $f(x,y)$ seja analítica

$$\frac{\partial v}{\partial x} = 2 - 2y \qquad \frac{\partial^2 v}{\partial x^2} = 0 \qquad \therefore \text{ É harmônica}$$

$$\frac{\partial v}{\partial y} = -2x \qquad \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial x} = -2x \qquad \therefore u = x^2 + \phi(y)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad -(2-2y) = \phi'(y) \qquad \therefore \phi(y) = -2y + y^2 + C$$

$f(x+yi) = 2xi - 2xyi + x^2 - 2y + y^2 + c$

$= -z^2 + 2zi$

$-z^2 = -(x^2 + xyi - y^2)$

$= -x^2 - xyi + y^2$

$z = x + yi$

$zi = xi - y$

(2) $\oint \dfrac{z^3 - z^2 + 2}{(z-1)^2(z-2)} dz$, onde $c|z| = 3$

$$\oint = \oint_{c1} \frac{(z^3-z^2+2).(z-1)^{-2}}{(z-2)} dz + \oint_{c2} \frac{(z^3-z^2+2)(z-2)^{-1}}{(z-1)^2} dz$$

C_1: $f(x) = \dfrac{z^3 - z^2 + 2}{(z-1)^2}$ $\qquad \oint_{c1} \dots = 2\pi i . 6 = 12\pi i$

$a = 2$

$n+1 = 1 \Rightarrow n = 0$

$f(2) = \dfrac{2^3 - 2^2 + 2}{(2-1)^2} = 6$

C_2: $f(x) = \dfrac{(z^3 - z^2 + 2)}{(z-2)}$ $\qquad f'(x) = \dfrac{\overset{1}{(3z^2 - 2z)}\overset{(-1)}{(z-2)} - (z^3 - z^2 + 2)}{z-2}$

$= \dfrac{-1 - (+2)}{-1} = \dfrac{-3}{-1} = 3$

$a = 1$

$n+1 = 2 \Rightarrow n = 1$

$I_2 = 2\pi i \cdot 3 = 6\pi i$ $\qquad I = 12\pi i + 6\pi i = \boxed{18\pi i}$

(3) $z^4 - 16i = 0$

$z^4 = 16i \rightarrow z = \sqrt[4]{16i}$

$16i \rightarrow (0, 16)$

$\rho = 16 \qquad \cos\theta = \frac{0}{16} = 0 \qquad \sin\theta = \frac{16}{16} = 1 \quad \therefore \theta = \frac{\pi}{2}$

$\rightarrow 16\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = 16 e^{i\pi/2}$

$$\sqrt[4]{z^4} = \sqrt[4]{16}\left(\cos\left(\frac{\frac{\pi}{2} + 2\pi k}{4}\right) + i \sin\left(\frac{\frac{\pi}{2} + 2\pi k}{4}\right)\right)$$

Para $k = 0$

$w_1 = 2\left(\cos\frac{\pi}{8} + i \sin\frac{\pi}{8}\right)$