Nome: Erivelton Gualter dos Santos 11.210.368-4

1º semestre de 2011 2º ciclo Engenharia Básica

Algebra linear (Parte I)

Álgebra linear

Espaço Vetorial $V = \emptyset$

Def.: $V \neq \emptyset$ é denominado espaço vetorial sobre $\mathbb{R} \Leftrightarrow$

(I) Existe uma adição em V tal que $\forall u, v \in V \longmapsto u + v \in V$ Satisfazendo as seguintes propriedades:

A1) Comutativa: $u + v = v + u$

A2) Associativa: $(u+v) + w = u + (v+w)$

A3) Elemento Neutro: $u + 0 = u$

A4) Elemento oposto: $u + (-u) = 0$

(II) Está definida a multiplicação de um nº real por um vetor.

$\forall \alpha \in \mathbb{R},\ \forall u \in V \longmapsto \alpha u \in V$

Propriedades:

M1) $(\alpha u)\beta = (\alpha\beta)u$

M2) $\alpha(u+v) = \alpha u + \alpha v$

M3) $(\alpha + \beta)u = \alpha u + \beta u$

M4) $1 . u = u$

$\vec{u} \neq \vec{0}$

$\vec{w} = 2\vec{u}$ $\qquad \alpha > 1$

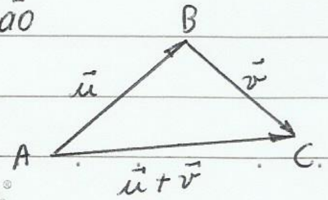
$\vec{w} = \frac{1}{2}\vec{u}$ $\qquad 0 < \alpha < 1$

$\vec{w} = -\frac{3}{2}\vec{u}$ $\qquad \alpha < 0$

Exemplos:

1º) Vetores Geométricos (G.A) (Segmentos Orientados)

Adição



Espaço Vetorial

2º Espaço vetorial das Matrizes do tipo $m \times n$ sobre $\mathbb{R}$: $V = M_{m \times n}(\mathbb{R})$

$A = (a_{ij})$; $A_{m \times n} \in V$

$B = (b_{ij})$; $B_{m \times n} \in V$

Def:

Adição: $A + B = (a_{ij}) + (b_{ij})$

Prod. p/ nº real: $\alpha A = \alpha (a_{ij})$

Propriedades

A1) Comutativa: $A + B = B + A$

A2) Associativa: $(A+B)+C = A+(B+C)$

A3) El. Neutro: $A + 0 = A$

A4) El. oposto: $A + (-A) = 0$

M1) $(\alpha A)\beta = (\alpha\beta)A$

M2) $\alpha(A+B) = \alpha A + \alpha B$

M3) $(\alpha + \beta)A = \alpha A + \beta A$

M4) $1 . A = A$

Exercício pág. 20

Dados os vetores

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 \\ 2 & 4 \\ -1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 0 & -1 \\ 2 & 3 \\ 4 & 2 \end{pmatrix}$$

Espaço Vetorial

Exemplos:

3º) $V = P_n(\mathbb{R})$: Espaço vetorial dos polinômios de grau $\leq n$ sobre $\mathbb{R}$, mais o polinomio nulo.

Dados

$p(t) : a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n \in P_n(\mathbb{R})$

$q(t) : b_0 + b_1 t + b_2 t^2 + \dots + b_n t^n \in P_n(\mathbb{R})$

Definimos:

Adição:

$p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1) t + (a_1 + b_2) t^2 + \dots + (a_n + b_n) t^n$

$= \sum_{i=0}^{n} (a_i + b_i) t^i \in P_n(\mathbb{R})$

Produto P/ nº Real:

$\alpha . p(t) = \alpha a_0 + \alpha a_1 t + \alpha a_2 t^2 + \dots + \alpha a_n t^n = \sum_{i=0}^{n} (\alpha a_i) t^i$

Propriedades:

A1) $p(t) + q(t) = q(t) + p(t)$

A2) $[p(t) + q(t)] + h(t) = p(t) + [q(t) + h(t)]$; $p(t), q(t), h(t) \in P_n(\mathbb{R})$

A3) $p(t) + 0(t) = p(t)$

A4) $p(t) + (-p(t)) = 0(t)$

M1) $(\alpha\, p(t))\beta = (\alpha\beta)(p(t))$; $\alpha, \beta \in \mathbb{R}$

M2) $\alpha(p(t) + q(t)) = \alpha\, p(t) + \alpha\, q(t)$

M3) $(\alpha + \beta)\, p(t) = \alpha\, p(t) + \beta\, p(t)$

M4) $1 . p(t) = p(t)$

4°) $V = \mathbb{R}^n$: Espaço vetorial dos vetores de n coordenadas (n-uplas) sobre $\mathbb{R}$

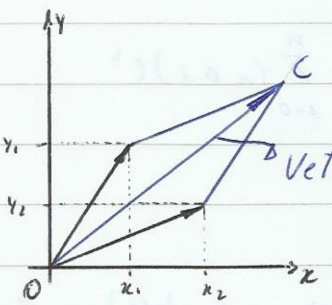
$u = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$
$v = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$
DEF:
$u + v = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$
$\alpha u = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$

Valem as 8 propriedades

Casos particulares

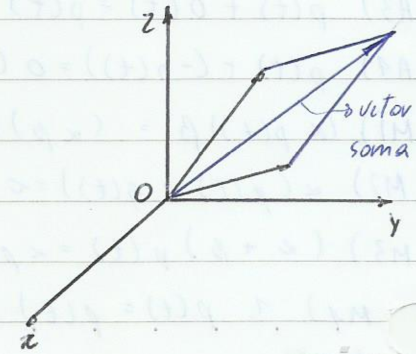
a) $n = 2$; $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$

Adição: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
prod. p/ nº real: $\alpha(x, y) = (\alpha x, \alpha y)$



b) $n = 3$, $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z)$



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Exercícios em sala

63-) $u=(1,1,1)$ $v=(0,-2,1)$ $w=(-1,0,1)$, dados os vetores, determine o vetor t: tal que:

$$\frac{t+u}{2}+\frac{v+t}{3}=w$$

$$\frac{(a,b,c)+(1,1,1)}{2}+\frac{(0,-2,1)+(a,b,c)}{3}=(-1,0,1) \Rightarrow$$

$$\Rightarrow \frac{a+1}{2}+\frac{a}{3}=-1 \Rightarrow \frac{3a+3+2a+6}{6}=0 \Rightarrow 5a=-9 \quad a=-9/5$$

$$\Rightarrow \frac{b+1}{2}+\frac{(-2)+b}{3}=0 \Rightarrow \frac{3b+3-4+2b}{6}=0 \quad 5b=1 \quad b=\frac{1}{5}$$

$$\Rightarrow \frac{c+1}{2}+\frac{1+c}{3}=1 \Rightarrow \frac{3c+3+2+2c-6}{6}=0 \quad 5c=1 \quad c=\frac{1}{5}$$

$$t=\left(\frac{-9}{5},\frac{1}{5},\frac{1}{5}\right)$$

$$\begin{cases} x+y+z=u \\ 2x+y+z=w \\ x+y+2z=v \end{cases} \sim \begin{array}{ccc|c} 1 & 1 & 1 & u \\ 2 & 1 & 1 & w \\ 1 & 1 & 2 & v \\ \hline & -1 & -1 & w-2u \\ & 0 & 1 & v-u \end{array}$$

$$\begin{cases} x+y+z=u \\ -y-z=w-2u \\ z=v-u \end{cases} \sim$$

$$\begin{cases} x = w-u \\ -y = w+v-3u \\ z=v-u \end{cases} \quad \begin{aligned} x&=(-1,0,1)-(1,1,1)=(-2,-1,0) \\ y&=(4,5,1) \\ z&=(0,-2,1)-(1,1,1)=(-1,-3,0) \end{aligned}$$

spiral

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64-
$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & -1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$$

a) $2A - 3B + C = \begin{bmatrix} 5 & 4 \\ 4 & -6 \\ -7 & 1 \end{bmatrix}$

b) $\dfrac{X-A}{2} + \dfrac{B-C}{3} = \dfrac{X+A-B}{4} \sim \dfrac{6(X-A)+4(B-C)=3(X+A-B)}{12}$

$6X - 6A + 4B - 4C - 3X - 3A + 3B = 0 \sim$

$3X - 9A + 7B - 4C = 0$

$X = \dfrac{9A - 7B + 4C}{3} = 9A - 7B + 4C$

$X = \begin{bmatrix} 22 & 12 \\ 11 & -15 \\ -12 & 5 \end{bmatrix}$

c) $\nexists \ \alpha, \beta \in \mathbb{R}$

65-
$f(t) = 2 - 4t + t^2 - t^3$
$g(t) = -1 + t - t^3$
$h(t) = 1 + 2t + t^2 + 2t^3$

a) $f(t) + 2g(t) - h(t) = 2 - 4t + t^2 - t^3 - 2 + 2t - 2t^3 - 1 - 2t - t^2 - 2t^3 =$
$= -1 - 4t - 5t^3$

b) $f(t) = \alpha g(t) + \beta h(t)$
$2 - 4t + t^2 - t^3 = \alpha(-1 + t - t^3) + \beta(1 + 2t + t^2 + 2t^3)$

$2 = -\alpha + \beta$ | $t^2 = \beta t^2$
$-4t = \alpha t + 2t\beta$ | $-t^3 = -\alpha t^3 + \beta 2 t^3$

spiral

$t^2 = \beta t^2 \quad \sim \quad \beta = 1$

$2 = -\alpha + \beta \quad \sim \quad \alpha = \beta - 2 \quad \sim \quad \alpha = 1 - 2 = -1$

$-t^3 = -\alpha t^3 + \beta 2 t^3 \quad \to \quad -t^3 \neq 1 t^3 + 2 t^3$

$\nexists \, \alpha, \beta \in \mathbb{R}$

Sub Espaço Vetorial

V: Espaço vetorial sobre $\mathbb{R}$

$W \subset V$ $(W \neq \emptyset)$

Proposição

W é subespaço de V

Se $\begin{cases} i) \ 0 \in W \text{ (vetor nulo)} \\ ii) \ \forall u, v \in W \Rightarrow u+v \in W \\ iii) \ \forall \lambda \in \mathbb{R}, \forall u \in W \Rightarrow \lambda v \in W \end{cases}$

Exemplos

1) $V = \mathbb{R}^3$

Verificar se algum dos subconjuntos dados, do $\mathbb{R}^3$ é subespaço

a) $W = \{(x, y, z) \in \mathbb{R}^3 \,/\, x + z \leq 0\}$

b) $W = \{(x, y, z) \in \mathbb{R}^3 \,/\, x + z \in \mathbb{Q}\}$

c) $W = \{(x, y, z) \in \mathbb{R}^3 \,/\, y^2 + z = 0\}$

d) $W = \{(x, y, z) \in \mathbb{R}^3 \,/\, x - 2z = 0\}$

a-) $W = \{(x, y, z) \in \mathbb{R}^3 \,/\, x + z \leq 0\}$

i) $0 = (0,0,0) \in W$ pois $0 + 0 \leq 0$

ii) $\left.\begin{array}{l} u = (x_1, \cancel{y_2}, \cancel{z_3}) \in \mathbb{R}^3 \,/\, x_1 + z_1 \leq 0 \\ v = (x_2, y_2, z_3) \in \mathbb{R}^3 \,/\, x_2 + z_2 \leq 0 \end{array}\right\}$ hipótese (H)

$u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \,/$

$(x_1 + x_2) + (z_1 + z_2) = \underbrace{(x_1 + z_1)}_{\leq 0} + \underbrace{(x_2 + z_2)}_{\leq 0} \leq 0$

$\therefore u + v \in W$

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iii) $\forall \lambda \in \mathbb{R}, \forall u \in W \,/\, x+z \leq 0$ Então

$\lambda u = (\lambda x, \lambda y, \lambda z) \,/\, \lambda x + \lambda z = \underbrace{\lambda}_{\mathbb{R}} \underbrace{(x+z)}_{\leq 0} \not\leq 0$

$\therefore \lambda u \notin W$

b) $W = \{(x,y,z) \in \mathbb{R}^3 \,/\, x+y \in \mathbb{Q}\}$

$\mathbb{Q} = \left\{ \frac{p}{q} ;\ p,q \in \mathbb{Z} \text{ e } q \neq 0 \right\}$

i) $0 = (0,0,0) \in W$ pois $0+0 \in \mathbb{Q}$

ii) $u = (x_1, y_1, z_1) \in W \,/\, x_1 + z_1 \in \mathbb{Q}$

$v = (x_2, y_2, z_2) \in W \,/\, x_2 + z_2 \in \mathbb{Q}$

$u+v = (x_1+x_2, y_1+y_2, z_1+z_2) \,/$

$(x_1+x_2) + (y_1+y_2) = \underbrace{(x_1+y_1)}_{\in \mathbb{Q}} + \underbrace{(x_2+y_2)}_{\in \mathbb{Q}} \in \mathbb{Q}$

$\therefore u+u \in W$

iii) $\forall \lambda \in \mathbb{R}, \forall u \in W \,/\, x+y \in \mathbb{Q}$ Então

$\lambda u = (\lambda x, \lambda y, \lambda z) \,/\, \lambda x + \lambda y = \underbrace{\lambda}_{\mathbb{R}} \underbrace{(x+y)}_{\in \mathbb{Q}} \notin \mathbb{Q}$

Exemplo: $\lambda = \sqrt{5}$

$(x+y) = \frac{7}{9}$ $\Rightarrow$ $\frac{7\sqrt{5}}{9} \notin \mathbb{Q}$

c) $W = \{(x,y,z) \in \mathbb{R}^3 \,/\, y^2 + z = 0\}$

i) $0 = (0,0,0) \in W$ pois $0^2 + 0 = 0$

ii) $\left.\begin{array}{l} u = (x_1 + y_1 + z_1) \in W \,/\, y_1^2 + z_1 = 0 \\ v = (x_2 + y_2 + z_2) \in W \,/\, y_2^2 + z_2 = 0 \end{array}\right\} H$

$u+v = (x_1+x_2, y_1+y_2, z_1+z_2) \,/$

?

$(y_1+y_2)^2 + (z_1+z_2) = 0 \Rightarrow = y_1^2 + 2y_1y_2 + y_2^2 + z_1 + z_2$

0 0

$\therefore u+v \notin W$

Proposição: Sejam $\mathcal{U}$ e $\mathcal{W}$ s.e. de V. Então $\mathcal{U} \cap \mathcal{W}$ é s.e. V.

hip: $\mathcal{U} \subset V$ é s.e
$\quad W \subset V$ é s.e
tese: $\mathcal{U} \cap W$ é s.e

i) $0 \in \mathcal{U}$, $0 \in \mathcal{W}$ $\Rightarrow$ $0 \in \mathcal{U} \cap \mathcal{W}$

ii) $u_1 \in \mathcal{U} \cap \mathcal{W} \Rightarrow \begin{cases} u_1 \in \mathcal{U} \\ u_1 \in W \end{cases}$ e

$\quad u_2 \in \mathcal{U} \cap \mathcal{W} \Rightarrow \begin{cases} u_2 \in \mathcal{U} \\ u_2 \in W \end{cases}$ e

Então

$\left.\begin{array}{l} u_1 + u_2 \in \mathcal{U} \\ u_1 + u_2 \in \mathcal{W} \end{array}\right\} u_1 + u_2 \in \mathcal{U} \cap \mathcal{W}$

iii) $u \in \mathcal{U} \cap \mathcal{W} \Rightarrow \left\{\begin{array}{l} u \in \mathcal{U} \Rightarrow \lambda u \in \mathcal{U},\ \forall \lambda \in \mathbb{R} \\ u \in W \Rightarrow \lambda u \in \mathcal{W},\ \forall \lambda \in \mathbb{R} \end{array}\right\} \Rightarrow \lambda u \in \mathcal{U} \cap \mathcal{W}$

Logo, $\mathcal{U} \cap \mathcal{W}$ é s.e de V

Ex. 1: Seja $V = \mathbb{R}^3$

Dados os s.e:

$\mathcal{U} = \{(x, y, z) \in \mathbb{R}^3 \mid z = x - y\}$ $\quad (x, y, x-y)$ $\begin{cases} x - y + z = 0 \\ z = 0 \end{cases}$

$\mathcal{W} = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$,

determine $\mathcal{U} \cap \mathcal{W}$ $\qquad x = y$

$(\mathcal{U} \cap \mathcal{W})$ é s.e de $\mathbb{R}^3$

Ex 2 $V = \mathbb{R}^3$

$\mathcal{U} = \{ (x,y,z) \in \mathbb{R}^3 \mid x = z \}$
$\mathcal{W} = \{ (0,0,z) \mid z \in \mathbb{R} \}$

$\mathcal{U} \cap \mathcal{W} = ?$

$$u = (x,y,z) \in \mathcal{U} \cap \mathcal{W} \Rightarrow \left\{ \begin{array}{l} (x,y,z) \in \mathcal{U} \Rightarrow x = z \\ (x,y,z) \in \mathcal{W} \Rightarrow x = y = 0 \end{array} \right\} \Rightarrow x = z = y = 0$$

logo, $\mathcal{U} \cap \mathcal{W} = \{ (0,0,0) \}$

Soma de subespaços

Seja V um e.v. sobre $\mathbb{R}$. Dados $\mathcal{U}$ e $\mathcal{W}$ s.e. de V,
$\mathcal{U} + \mathcal{W} = \{ u + w \mid u \in \mathcal{U} \text{ e } w \in \mathcal{W} \}$ é chamado <u>soma de $\mathcal{U}$ com $\mathcal{W}$</u>.

Proposição: $\mathcal{U} + \mathcal{W}$ é s.e de V

Prova: i) $0 \in \mathcal{U} + \mathcal{W}$, pois

$$0 = \underset{\mathcal{U}}{0} + \underset{\mathcal{W}}{0}$$

ii) Sejam $u, v \in \mathcal{U} + \mathcal{W}$

$u \in \mathcal{U} + \mathcal{W} \Rightarrow u = \underset{\in \mathcal{U}}{u_1} + \underset{\in \mathcal{W}}{w_1}$

$v \in \mathcal{U} + \mathcal{W} \Rightarrow v = \underset{\in \mathcal{U}}{u_2} + \underset{\in \mathcal{W}}{w_2}$

$u + v = (u_1 + w_1) + (u_2 + w_2) = \underset{\in \mathcal{U}}{(u_1 + u_2)} + \underset{\in \mathcal{U}}{(w_1 + w_2)}$

$= u_3 + w_3$

iii) $z \in \mathcal{U}+\mathcal{W} \Rightarrow z = u + w$, $u \in \mathcal{U}$ e $w \in \mathcal{W}$

$\Rightarrow \lambda z = \underbrace{\lambda u}_{\in \mathcal{U}} + \underbrace{\lambda w}_{\in \mathcal{V}} \in \mathcal{U}+\mathcal{W}, \forall \lambda \in \mathbb{R}$

$\mathcal{U}+\mathcal{W}$ é s.e

Ex1 $V = \mathbb{R}^3$

$\mathcal{U} = \{(x,y,z) \in \mathbb{R}^3 \mid z = x - y\}$

$W = \{(x,y,0) \mid x, y \in \mathbb{R}\}$

Obs: $A = B \Leftrightarrow \begin{cases} i) \ A \subset B \ \text{e} \\ ii) \ B \subset A \end{cases}$

Obs 1: $\left.\begin{array}{l} \mathcal{U} \underset{s.e}{\subset} V \\ \mathcal{W} \underset{s.e}{\subset} V \end{array}\right\} \Rightarrow \mathcal{U}+\mathcal{W} \underset{s.e}{\subset} V$

a) $\mathcal{U}+\mathcal{W} \subset V$

b) $V \subset \mathcal{U}+\mathcal{W}$

$v = (x,y,z) \in \mathbb{R}^3$

$v = (x,y,z) = \underbrace{(y+z, y, z)}_{\in \mathcal{U}} + \underbrace{(x-y-z, 0, 0)}_{\in \mathcal{W}}$

$\therefore v \in \mathcal{U}+\mathcal{W}$

(a) e (b) $\Rightarrow \mathcal{U}+\mathcal{W} = \mathbb{R}^3$

Ex2 $V = \mathbb{R}^3$

$\mathcal{U} = \{(x,y,z) \in \mathbb{R}^3 \mid x = z\}$

$\mathcal{W} = \{(0,0,z)\ z \in \mathbb{R}\}$

a) $\mathcal{U}+\mathcal{W} \underset{se}{\subset} \mathbb{R}^3$

$v = (x,y,z) = \underbrace{(x,y,x)}_{\mathcal{U}} + \underbrace{(0,0,z-x)}_{\mathcal{W}}$

b) $\mathbb{R}^3 \subset \mathcal{U}+\mathcal{W}$

$\therefore \mathbb{R}^3 \subset \mathcal{U}+\mathcal{W}$

(a) e (b) $\Rightarrow \mathbb{R}^3 = \mathcal{U}+\mathcal{W}$

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Soma Direta

Seja V e.v. sobre $\mathbb{R}$

Def: Dados $\mathcal{U}$ e $\mathcal{W}$ s.e. de V, dizemos que V é soma direta de $\mathcal{U}$ com $\mathcal{W}$, $\boxed{V = \mathcal{U} \oplus \mathcal{W}}$, se:

i) $\mathcal{U} \cap \mathcal{W} = \{0\}$

ii) $\mathcal{U} + V = V$

No ex.1. $\mathbb{R}^3$ não é soma direta de $\mathcal{U}$ com $\mathcal{W}$ é uma reta. No ex.2. $\mathbb{R}^3 = \mathcal{U} \oplus \mathcal{W}$ pois, como já vimos,

i) $\mathcal{U} \cap \mathcal{W} = \{(0,0,0)\}$ e

ii) $\mathcal{U} + \mathcal{W} = \mathbb{R}^3$

Proposição: $\mathcal{U} \oplus \mathcal{W} = V \iff$ cada $v \in V$ se decompõe de maneira única como $v = u + w$, onde $u \in \mathcal{U}$ e $w \in \mathcal{W}$

Combinações lineares (C.L.)

Dados os vetores do esp. vetorial $V=\mathbb{R}^3$, $u=(1,1,1)$, $v=(-1,2,3)$ e $w=(2,1,5)$ 1-) Verificar se $\exists\ \alpha, \beta \in \mathbb{R} \mid w = \alpha u + \beta v$

2-) Interpretar analiticamente

3-) Dar a eq. homogenea do subespaço que u e v determinam.
(Obs.: Sem determinante)

1-) $w = \alpha u + \beta v$

$(2,1,5) = \alpha(1,1,1) + \beta(-1,2,3);\ \alpha, \beta \in \mathbb{R}$

$$\begin{cases} \alpha - \beta = 2 \\ \alpha + 2\beta = 1 \\ \alpha + 3\beta = 5 \end{cases} \qquad \begin{cases} \alpha - \beta = 2 \quad (\times -1) \\ \alpha + 2\beta = 1 \end{cases} \qquad \begin{cases} -\alpha + \beta = -2 \\ \alpha + 2\beta = 1 \end{cases}$$

$3\beta = -1$

$\beta = -1/3$

$\alpha = 1 - 2\beta$

$\alpha = 1 - 2.\left(\frac{-1}{3}\right)$

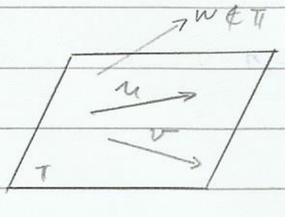
$\alpha = 5/3$

$\alpha + 3\beta = 5$

$\frac{5}{3} + 3.\left(\frac{-1}{3}\right) = \frac{2}{3} \neq 5$

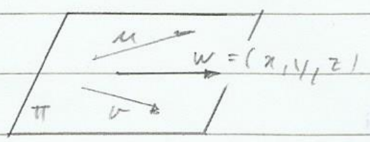
R: Não existe combinação linear entre u e v (S.I)

2-)



u, v e w não são coplanares

3-) Eq. homogênea do subespaço



$\forall w \in \pi \Rightarrow w = \alpha u + \beta v$

$(x, y, z) = \alpha(1,1,1) + \beta(-1,2,3)$

$$\begin{cases} \alpha - \beta = x \\ \alpha + 2\beta = y \\ \alpha + 3\beta = z \end{cases} \sim \begin{cases} \alpha - \beta = x \\ 3\beta = y - x & (\times 4) \\ 4\beta = z - x & (-3) \end{cases} \sim \begin{cases} \alpha - \beta = x \\ 3\beta = y - x \\ 0 = -x + 4y - 3z \end{cases}$$

$\pi: -x + 4y - 3z$ ou $x - 4y + 3z = 0$

R: $S = \{(x, y, z) \in \mathbb{R}^3 \mid x - 4y + 3z = 0\}$

Definição

V: Espaço vetorial sobre $\mathbb{R}$

$S = \{u_1, u_2, \ldots, u_n\} \subset V$

Indicamos por $[S]$ o seguinte subconjunto de V

$[S] = \{\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \mid \alpha_i \in \mathbb{R}\}$ $i = 1, 2, \ldots, n$

$[S]$ = é subespaço

i) $0 \in [S]$ pois $0 = 0.u_1 + 0.u_2 + \ldots + 0.u_3$

ii) $\forall u, v \in [S] \Rightarrow u + v \in [S]$

$\forall u \in [S] \Rightarrow u = \sum_{i=1}^{n} \alpha_i u_i$

$\forall v \in [S] \Rightarrow v = \sum_{i=1}^{n} \beta_i \cancel{v_i}$

$u + v \in [S] \mid u + v = \sum \alpha_i u_1$; $\alpha_i, \beta_i \in \mathbb{R}$

$= \sum_{i=1}^{n} \underbrace{(\alpha_i + \beta_i)}_{\gamma_i} u_i = \sum_{i=1}^{n} \gamma_i u_i$

iii) $\forall \lambda \in \mathbb{R}$, $\forall u \in [S] \mid u = \sum_{i=1}^{n} \alpha_i \cancel{u_i}$

Então $\lambda u = \lambda \sum \alpha_i u_i = \sum_{i=1}^{n} \underbrace{(\alpha_i \lambda)}_{\delta_i} u_i = \sum_{i=1}^{n} \delta_i u_i$

Def:

$[S]$ é denominado subespaço <u>gerado</u> por S. Cada vetor de $[S]$ é uma C.L. de S.

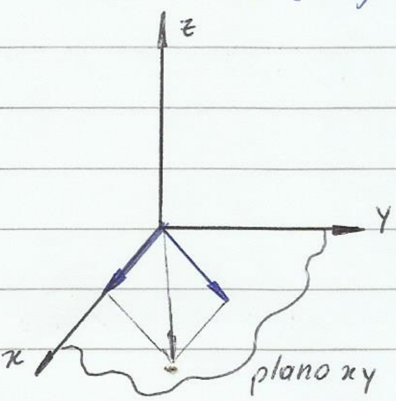
Dizemos ainda que os vetores $u_1, u_2, \ldots, u_n$ constituem um sistema de geradores e indicamos $[S] = [u_1, u_2, \ldots, u_n]$

Combinações lineares (C.L)

Exemplo:

$V = \mathbb{R}^3$ Dados os vetores $u = (1,0,0)$ e $v = (1,1,0)$ o que é $[u,v]$?

"Qual é o subespaço gerado por u e v"



$$\begin{aligned}[u,v] &= \{\alpha u + \beta v \mid \alpha, \beta \in \mathbb{R}\} \\ &= \{\alpha(1,0,0) + \beta(1,1,0) \mid \alpha, \beta \in \mathbb{R}\} \\ &= \{(\alpha+\beta, \beta, 0) \mid \alpha, \beta \in \mathbb{R}\} \\ &= \{(x, y, 0) \mid x, y \in \mathbb{R}\}\end{aligned}$$

onde $\alpha + \beta = x$, $\beta = y$

$V = \mathbb{R}^3$

Dado o subespaço do $\mathbb{R}^3$

$T = \{(x,y,z) \mid x - 2y = 0\}$

Dar um sistema de geradores de T.

$$x - 2y = 0 \Rightarrow x = 2y$$

$$\begin{aligned}T \ni (x,y,z) = (2y, y, z) &= \underbrace{(2y, y, 0)} + \underbrace{(0,0,z)} \\ &= y\underbrace{(2,1,0)}_{u} + z\underbrace{(0,0,1)}_{v}\end{aligned}$$

R: $[T] = [(2,1,0), (0,0,1)]$

/ /

$V = \mathbb{R}^3$

Dados os subespaços S e T (exemplos anteriores) Dar um sistema de gerações de $S \cap T$

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x - 4y + 3z = 0\}$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y = 0\}$$

$$S \cap T = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y = 0 \text{ e } x - 4y + 3z = 0\}$$

$$\begin{cases} x - 2y = 0 \\ x - 4y + 3z = 0 \end{cases} \qquad \begin{matrix} x = 2y \\ 2y - 4y + 3z = 0 \Rightarrow 3z - 2y = 0 \Rightarrow \\ z = \frac{2}{3}y \end{matrix}$$

$$S \cap T \ni (x, y, z) = (2y, y, \tfrac{2}{3}y)$$
$$= y(2, 1, 2/3)$$

$$S \cap T = [(2, 1, 2/3)]$$

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Combinações Lineares - geradores

Exercícios

$\boxed{76}$ página 50

$V = P_3(\mathbb{R})$

Determine $m \in \mathbb{R}$ para que o vetor $p(x) = 2 - x + mx^2 + x^3$ Seja C.L dos vetores

$f(x) = 1 - x - x^2$

$g(x) = 1 - 2x + x^2 - x^3$

$h(x) = 1 + x + x^2 + x^3$

86) Determine o gerador de $U \cap T$ sendo $U = [(1,0,0),(2,1,1)]$ e $T = [(1,-1,2),(0,1,-1)]$

87) Os vetores $(1,-1,3)$, $(0,1,-1)$ e $(2,3,1)$ Geram o $\mathbb{R}^3$?
(Obs.: sem determinante)

88) Dados os subespaços do $\mathbb{R}^3$

$u = \{(x,y,z) \mid x + 2y + 3z = 0\}$ e

$w = \{ \quad " \quad \mid 3x - y - z = 0\}$

Dar um sistema de geradores de $U, W, U \cap W$ e $U + W$

76.) $p(x) = af(x) + bg(x) + ch(x)$

$2 - x + mx^2 + x^3 = a(1 - x - x^2) + b(1 - 2x + x^2 - x^3) + c(1 + x + x^2 + x^3)$

$2 - x + mx^2 + x^3 = a - ax - ax^2 + b - 2bx + bx^2 - bx^3 + c + cx + cx^2 + cx^3$

$$\begin{cases} a - b + c = 2 \\ -ax - 2bx + cx = -x \\ -ax^2 + bx^2 + cx^2 = mx^2 \\ -bx^3 + cx^3 = x^3 \end{cases}$$

/ /

$2-x-mx^2+x^3=(a-b+c)+(-a-2b+c)x+(-a+b+c)x^2+(-b+c)x^3$

$$\begin{cases} a-b+c=2 \\ -a-2b+c=-1 \\ -a+b+c=m \\ -b+c=1 \end{cases} \sim \begin{cases} a-b+c=2 \\ -a-2b+c=-1 \\ -b+c=1 \\ -a+b+c=m \end{cases} \mathrm{I}$$

$$\begin{cases} a-b+c=2 \\ -3b+2c=1 \\ -b+c=1 \end{cases} \sim \begin{cases} a-b+c=2 \\ -3b+2c=1 \\ -c=-2 \end{cases}$$

$(a,b,c)=(1,1,2)$

$-a+b+c=m$

$-1+1+2=m \Rightarrow \underline{m=2}$

86-) $V=\mathbb{R}^3$

$U=[(1,0,0),(1,1,1)]$

$\forall w \in U$ é C.L dos vetores de U

$w=(x,y,z)=\alpha(1,0,0)+\beta(1,1,1)$; $\alpha,\beta \in \mathbb{R}$

$$\begin{cases} \alpha+\beta=x \\ \beta=y \\ \beta=z \end{cases} \sim \begin{cases} \alpha+\beta=x \\ \beta=y \\ \underline{0=z-y} \end{cases}$$

$T=[(1,-1,2),(0,1,-1)]$.

$\forall w \in T$ é CL dos vetores de T

$(x,y,z)=\alpha(1,-1,2)+\beta(0,1,-1)$; $\alpha,\beta \in \mathbb{R}$

$$\begin{cases} \alpha=x \\ -\alpha+\beta=y \\ 2\alpha-\beta=z \end{cases} \sim \begin{cases} \alpha=x \\ \beta=y+x \\ -\beta=z-2x \end{cases} \sim \begin{cases} \alpha=x \\ \beta=y+x \\ \underline{0=-x+y+z} \end{cases}$$

$U \cap T$

$\begin{cases} -y+z=0 & \Rightarrow z=y \\ -x+y+z=0 & \Rightarrow x=2y \end{cases}$

$(x,y,z) \in U \cap T /$

$(x,y,z) = (2y, y, y)$

$= y(2,1,1)$

R: $U \cap T = [(2,1,1)]$

87-) $(1,-1,3) = a(0,1,-1) + b(2,3,1)$

$\begin{cases} 2b = 1 \\ a+3b=-1 \\ -a+b=3 \end{cases}$ $b = \frac{1}{2}$ $a + 3 \cdot \frac{1}{2} = -1 \Rightarrow a = -1 - \frac{3}{2} = \frac{-2-3}{2} = -\frac{5}{2}$

$-a+b=3$

$\frac{5}{2} + \frac{1}{2} = \frac{6}{2} = 3 = 3$

$\therefore$ Os três vetores são coplanares, portanto eles não são $\mathbb{R}^3$

$w = (x,y,z) = a(1,-1,3) + b(0,1,-1) + c(2,3,1)$

$\begin{cases} a+2c=x \\ -a+b+3c=y \\ 3a-b+c=z \end{cases} \sim \begin{cases} -a+b+3c=y \\ b+5c=x+y \\ 2b+10c=3y+z \end{cases} \sim \begin{cases} -a+b+3c=y \\ b+5c=x+y \\ 0=-2x+y+z \end{cases}$

Por este método, forma uma equação do plano, portanto, os vetores são coplanares, com isto eles não são $\mathbb{R}^3$

88-) $U = \{ (x,y,z) \mid x + 2y + 3z = 0 \}$ e
$W = \{ \quad " \quad \mid 3x - y - z = 0 \}$

Gerador de U

$x + 2y + 3z = 0 \quad \therefore \quad x = -2y - 3z$

$\forall (x,y,z) \in U \,/$

$(x,y,z) = (-2y-3z, y, z)$
$= (-2y, y, 0) + (-3z, 0, z)$
$= y(-2, 1, 0) + z(-3, 0, 1)$

$U = [(-2,1,0), (-3,0,1)]$

Gerador
$U+W$
$[(-),(-),(-),(-)]$

Gerador de W

$3x - y - z = 0 \quad \therefore \quad z = 3x - y$

$\forall (x,y,z) \in W \,/$

$(x,y,z) = (x, y, 3x-y)$
$= (x, 0, 3x) + (0, y, -y)$
$= x(1, 0, 3) + y(0, 1, -1)$

$W = [(1,0,3), (0,1,-1)]$

$U \cap W$

$$\begin{cases} x + 2y + 3z = 0 \\ 3x - y - z = 0 \end{cases} \sim \begin{cases} x + 2y + 3z = 0 \\ -7y - 10z = 0 \end{cases} \quad \therefore$$

Dependência linear dos vetores

Seja V um espaço vetorial sobre $\mathbb{R}$

Def: Dizemos que um conjunto $\{v_1, v_2, \dots v_n\}$ de vetores de V é linearmente dependente (L.d) se existirem escalares $\alpha_1, \alpha_2, \dots, \alpha_n$, não todos nulos, tais que $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \bar{o}$ ($\bar{o}$: vetor nulo de V).

Caso contrário, isto é, se $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \bar{o} \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ o conjunto $\{v_1, v_2, \dots, v_n\}$ é dito linearmente independente (L.I)

Ex.1 V: vetores da Geometria Analítica

Ex.2: $V = \mathbb{R}^3$, dado o conjunto de vetores:

$\{\vec{v_1} = (1,2,0);\ \vec{v_2} = (-1,0,1);\ \vec{v_3} = (1,2,3)\}$

verifique se ele é LI ou LD

$\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \alpha_3 \vec{v_3} = (0,0,0)$

$\alpha_1 (1,2,0) + \alpha_2 (-1,0,1) + \alpha_3 (1,2,3) = (0,0,0)$

$(\alpha_1 - \alpha_2 + \alpha_3,\ 2\alpha_1 + 2\alpha_3,\ \alpha_2 + 3\alpha_3) = (0,0,0)$

$$\begin{cases} \alpha_1 - \alpha_2 + \alpha_3 = 0 \\ 2\alpha_1 + 2\alpha_3 = 0 \\ \alpha_2 + 3\alpha_3 = 0 \end{cases} \sim \begin{cases} \alpha_1 - \alpha_2 + \alpha_3 = 0 \\ 2\alpha_1 + 2\alpha_3 = 0 \\ \alpha_2 + 3\alpha_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Como $\begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = 2 - 2 + 6 = 6 \neq 0 \Rightarrow$ S.P.D Logo, $\{\bar{v_1}, \bar{v_2}, \bar{v_3}\}$ é L.I

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{bmatrix}$$

Obs: Para testar a dependência linear de vetores do $\mathbb{R}^n$, podemos escalonar a matriz formada pelas coordenadas desses vetores e, se a matriz escalonada apresentar alguma linha nula o conjunto é L.D; caso contrário o conjunto é LI.

Ex 2: $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4} = (2,2,-2)\}$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ é l.d

Ex 3: $V = \mathbb{R}^5$

$\{v_1 = (1,-1,0,2,1)\ ;\ v_2 = (2,0,0,1,2)\ ;\ v_3 = (0,1,-1,0,0)\}$

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 2 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix} \quad \therefore \{v_1, v_2, v_3\} \text{ é L.I}$$

Ex. 4 $V = P^2(t) \Rightarrow a_0 + a_1 t + a_2 t^2$

$\{ p_1(t) = 1 ; p_2(t) = 2+t ; p_3(t) = t + 2t^2 \}$ é li ou ld?

$\alpha_1 p_1(t) + \alpha_2 p_2(t) + \alpha_3 p_3(t) = \bar{0}$

$\alpha_1 . 1 + \alpha_2 (2+t) + \alpha_3 (t + 2t^2) = 0 + 0.t + 0.t^2$

$(\alpha_1 + 2\alpha_2) + (\alpha_2 + \alpha_3)t + 2\alpha_3 t^2 = 0 + 0.t + 0.t^2$

$$\Leftrightarrow \begin{cases} \alpha_1 + 2\alpha_2 = 0 \Rightarrow \alpha_1 = 0 \\ \alpha_2 + \alpha_3 = 0 \\ 2\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \end{cases} \Rightarrow \alpha_2 = 0$$

Logo, $\{ p_1(t), p_2(t), p_3(t) \}$ é L.I

Ex. 5: $V = M_2(\mathbb{R})$

$$\left\{ A_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} ; A_2 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} ; A_3 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} ; A_4 = \begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix} \right\}$$ é LI ou LD?

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\alpha_1 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 + 2\alpha_4 & \alpha_1 + \alpha_3 + 2\alpha_4 \\ -\alpha_1 + 2\alpha_2 + \alpha_3 - 2\alpha_4 & 2\alpha_2 - \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Resolução no verso

$$\Leftrightarrow \begin{cases} \alpha_1 + 2\alpha_4 = 0 \Rightarrow \alpha_1 = -2\alpha_4 \\ \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = 0 \\ -\alpha_1 + 2\alpha_2 + \alpha_3 - 2\alpha_4 = 0 \\ -\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \end{cases}$$

$\alpha_1 + 2\alpha_2 + 2\alpha_4 = \cancel{-2\alpha_4} + 2\alpha_2 + \cancel{2\alpha_4} = 0$

$\alpha_2 = 0$

$-\alpha_1 + 2\alpha_2 - 2\alpha_4 = 2\alpha_4 + 2.0 + 0 - 2\alpha_4 = 0$

S.P.I $\Rightarrow \{\ \}$ L.d

<u>Exercício</u> : $V = \{f: \mathbb{R} \to \mathbb{R}\}$ — verificar

$\{f(t) = 1 ; g(t) = e^t ; h(t) = e^{2t}\}$, $t \in \mathbb{R}$

$f \in V$

$f: \mathbb{R} \to \mathbb{R}$

$t \to f(t) = 0$

$\alpha_1 f(t) + \alpha_2 g(t) + \alpha_3 h(t) = \bar{0}, \forall t \in \mathbb{R}$

$\alpha_1 . 1 + \alpha_2 . e^t + \alpha_3 e^{2t} = \bar{0}, \forall t \in \mathbb{R}$

Admitindo $t = 0$

$\alpha_1 + \alpha_2 . e^0 + \alpha_3 e^{2.0} = 0$

$\alpha_1 + \alpha_2 + \alpha_3 = 0$

Admitindo $t = 1$

$\alpha_1 + \alpha_2 e^1 + \alpha_3 e^{2.1} = \bar{0}(1) = 0$

$\alpha_1 + \alpha_2 e + \alpha_3 e^2 = 0$

$\bar{0}: \mathbb{R} \to \mathbb{R}$

$t \mapsto \bar{0}(t) = 0$

Admitindo $t = 2$

$\alpha_1 + \alpha_2 e^2 + \alpha_3 e^4 = \bar{0}(2) = 0$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 e + \alpha_3 e^2 = 0 \\ \alpha_1 + \alpha_2 e^2 + \alpha_3 e^4 = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & e & e^2 \\ 1 & e^2 & e^4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & e-1 & e^2-1 \\ 0 & e^2-1 & e^4-1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & e-1 & e^2-1 \\ 0 & 0 & -e^4+e^3+e^2-e \end{bmatrix} \Rightarrow \text{S.P.D}$$

$\neq 0$

$\therefore$ Conjunto é L.I

Outra maneira é derivar a função

$\alpha_1 f(t) + \alpha_2 g(t) + \alpha_3 h(t) = \bar{0}$

(HOMEWORK)

Proposição:

Sejam: V e.v. real e $S = \{v_1, v_2, \dots, v_n\}$ conj. de vetores de V.

1) Se S é L.I, qualquer $S_1 \subset S$ (S_1 subconjunto de S) é l.i

2) Se um subconjunto S_2 de S ($S_2 \subset S$) é l.d, então S é l.d.

3) S é l.d $\iff$ um deles se exprime como combinação linear dos outros.

/ /

Dependencia Linear Base e Dimensão

Exercícios $V = \mathbb{R}^3$

- Determine $m \in \mathbb{R}$ para que seja L.I o seguinte subconjunto do $\mathbb{R}^3$:

$\{(m,1,2), (1,1,3), (0,-1,1)\}$

Obs.: Sem determinante

Aplicar o processo prático matricial

- $V = \mathbb{R}^4$ - Determine $m \in \mathbb{R}$ para que seja L.D. o seguinte subconjunto do $\mathbb{R}^4$.

$\{(1,2,1,0), (2,m,0,0), (1,3,3,0)\}$

- $V = P_2(\mathbb{R})$

$S = \{1-x,\ 1-x-x^2,\ 2x+x^2\}$

S é LI ou LD

$$\begin{pmatrix} m & 1 & 2 \\ 1 & 1 & 3 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ m & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1-m & 2-3m \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 3-4m \end{pmatrix}$$

$3-4m \neq 0 \quad m \neq 3/4$

Para determinar um subconjunto do LI a última linha deve ser $\neq 0$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \\ 2 & m & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & m-4 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 6-2m & 0 \end{pmatrix}$$

$6-2m = 0$

$m = 3 \rightarrow$ p/ ser LD

$\rightarrow \quad S = \{1-x, \ 1-x-x^2, \ 2x+x^2\}$

$$\alpha(1-x) + \beta(1-x-x^2 + \gamma(2x+x^2) = 0 + 0x + 0x^2$$

$$\begin{cases} \alpha + \beta = 0 \\ -\alpha x - \beta x + 2x\gamma = 0 \\ -\beta x^2 + x^2\gamma = 0 \end{cases} \qquad \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$\therefore$ V é L.I.

Base e Dimensão

(I) Base de um espaço vetorial finitamente gerado

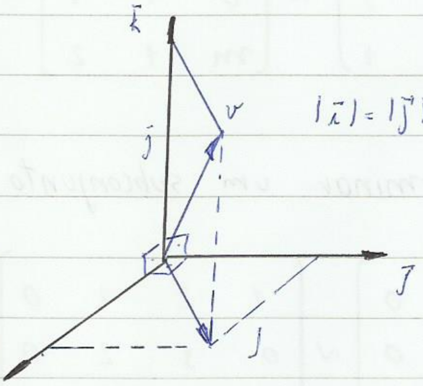
V: Espaço vetorial sobre $\mathbb{R}$

$B \subset V$, $B \neq 0$

Def:

B é denominado base de V

$\Leftrightarrow \begin{cases} (i) \ B \text{ é L.I} \\ (ii) \ V = [B] \end{cases}$ Conjunto gerador



$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$

Exemplos:

1) G.A

$B = \{\vec{i}, \vec{j}, \vec{k}\}$

B é base:

i) B é LI

$\alpha\vec{i} + \beta\vec{j} + \gamma\vec{k} = \vec{0} \Leftrightarrow \alpha = \beta = \gamma = 0$

ii) B é gerador

$\forall v = (x, y, z)$ é C.L de B

$v = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k} \qquad v = (3, 4, 5) = 3\vec{i} + 4\vec{j} + 5\vec{k}$

Base e Dimensão

2) $V = \mathbb{R}^2$; $B = \{(1,0), (0,1)\}$

i) B é L.I

$\alpha(1,0) + \beta(0,1) = (0,0) \Rightarrow \alpha = \beta = 0$

ii) B é gerador

$\forall (x,y) \in \mathbb{R}^2$ é C.L de B

$(x,y) = a(1,0) + b(0,1)$

$$\begin{cases} a = x \\ b = y \end{cases}$$

$(7,9) = 7(1,0) + 9(0,1)$

3) $V = \mathbb{R}^3$; $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

B é base do $\mathbb{R}^3$

i) B é L.I

$\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) = (0,0,0)$

$\therefore \alpha = \beta = \gamma = 0$

ii) B é Gerador

$\forall (x,y,z) \in \mathbb{R}^3$ é C.L de B:

$(x,y,z) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1)$

$$\begin{cases} \alpha = x \\ \beta = y \\ \gamma = z \end{cases}$$

$(3,2,7) = 3(1,0,0) + 2(0,1,0) + 7(0,0,1)$

/ /

4) $V = \mathbb{R}^4$

$B = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

B é base do $\mathbb{R}^4$

i) B é L.I

$\alpha(1,0,0,0) + \beta(0,1,0,0) + \gamma(0,0,1,0) + \lambda(0,0,0,1) = (0,0,0,0)$

$\alpha = \beta = \gamma = \lambda = 0$

ii) B é gerador: $\forall (x,y,z,t) \in \mathbb{R}^4$ é C.L de B.

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} a = x \\ b = y \\ c = z \\ d = t \end{cases}$$

ii) B é gerador: $\forall (x,y,z,t) \in \mathbb{R}^4$ é C.L de B

$(x,y,z,t) = \alpha(1,0,0,0) + \beta(0,1,0,0) + \gamma(0,0,1,0) + \lambda(0,0,0,1)$

$$\begin{cases} \alpha = x \\ \beta = y \\ \gamma = z \\ \lambda = t \end{cases}$$

spiral

5) $V = M_{2\times 2}(\mathbb{R})$

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

i) B é L.I

$$\alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \alpha = \beta = \gamma = \lambda = 0$

ii) $\forall \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ é C.L de B

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} a = x \\ b = y \\ c = z \\ d = t \end{cases}$$

6) $V = P_n(\mathbb{R})$

$B = \{1, t, t^2 \ldots, t^n\}$

i) B é L.I

$\alpha_0 . 1 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n = 0$

$\therefore \alpha_0 = \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

$\alpha_i \in \mathbb{R}$

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ii) B é gerador

$\forall f(t) \in P_n(\mathbb{R})$ é CL de B:

$f(t) = a_0 \cdot 1 + a_1 \cdot t + a_2t^2 + \dots + a_nt^n$

ou

$f(t) = a_0f_0 + a_if_i + a_2f_2 + \dots + a_nf_n$

onde: $f_0 = 1$, $f_1 = t$, $f_2 = t^2$, ..., $f_n = t^n$

Dimensão de um espaço vetorial finitamente gerado

Teorema da Invariância

Seja V um espaço vetorial finitamente gerado. Então, 2 bases quaisquer de V tem o mesmo número de vetores

Definição: Dimensão de um espaço vetorial V finitamente gerado é o nº de vetores de qualquer uma das bases de V

Exemplos:

dim $\mathbb{R}^2 = 2$

" $\mathbb{R}^3 = 3$

" $\mathbb{R}^4 = 4$

" $\mathbb{R}^n = n$

" $M_{2x2}(\mathbb{R}) = 4$

" $M_{3x3}(\mathbb{R}) = 9$

" $M_{nxn}(\mathbb{R}) = n^2$

" $M_{mxn}(\mathbb{R}) = mn$

" $P_n(\mathbb{R}) = n+1$

Exercício Base e Dimensão

Dar uma base e a dimensão do seguinte subespaço do $\mathbb{R}^4$:

$$W = \{(x,y,z,t) \mid x - 2y = z + 3t = 0\}$$

$$\begin{cases} x - 2y = 0 \Rightarrow x = 2y \\ z + 3t = 0 \Rightarrow z = -3t \end{cases}$$

$(x,y,z,t) \in W \mid$

$$(x,y,z,t) = (2y, y, -3t, t)$$
$$= (2y, y, 0, 0) + (0, 0, -3t, t)$$
$$= y(2,1,0,0) + t(0,0,-3,1)$$

Base $W = \{(2,1,0,0), (0,0,-3,1)\}$

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 \end{pmatrix} \qquad \dim W = 2$$

Base e dimensão

Def: Dizemos que o conjunto $\{v_1, v_2, \dots, v_n\}$ de vetores de V formam uma base de V se:

i) $\{v_1, v_2, \dots, v_n\}$ é L.I

ii) $[v_1, v_2, \dots, v_n] = V$

Ex: $V = \mathbb{R}^2$ $\{v_1 = (1,-1)\}$

$v = -5v_1$: v é C.L de v_1

i) $S = [(1,-1)] = \{\alpha(1,-1) \mid \alpha \in \mathbb{R}\}$ ii) $\alpha(1,-1) = (0,0) \iff \alpha = 0$ $\{(1,-1)\}$ é L.I

Não, pois, p. ex. $v = (-3,2)$ não é c.l de $(1,-1)$: $\nexists \alpha \in \mathbb{R} \mid (-3,2) = \alpha(1,-1) = (\alpha, -\alpha)$

$B_2 = \{v_1 = (1,-1) ; v_2 = (2,1)\}$

$\begin{cases} \alpha = 3 \\ \alpha = 2 \end{cases}$ absurd, logo, $\{(1,-1)\}$ não é base do $\mathbb{R}^3$

i) $[(1,-1), (2,1)] = \{\alpha_1(1,-1) + \alpha_2(2,1) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$

i) $\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ $\therefore$ S é L.I

ii) a) $[S] = [(1,-1), (2,1)] \subset \mathbb{R}^2$

b) $\mathbb{R}^2 \subset [S]$

$\forall (x,y) \in \mathbb{R}^2$, existem nºs reais α_1, α_2 tais que $(x,y) = \alpha_1(1,-1) + \alpha_2(2,1)$

$$\begin{cases} x = \alpha_1 + 2\alpha_2 \\ y = -\alpha_1 + \alpha_2 \end{cases} \Rightarrow x + y = 3\alpha_2 \quad \therefore \quad \alpha_2 = \frac{x+y}{3}$$

$$\alpha_1 = x - 2\alpha_2$$
$$= x - 2\left(\frac{x+y}{3}\right) = \frac{1}{3}x - \frac{2}{3}y$$

$\therefore \mathbb{R}^2 = [S]$

Teorema:

Sejam: V e.v. de dimensão n. U e W subespaços vetoriais de V. Então,

$$\boxed{\dim(U+W) = \dim U + \dim W - \dim(U \cap W)}$$

Exercício Base e Dimensão

Dar uma base e a dimensão do seguinte subespaço do $\mathbb{R}^4$:

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$$\begin{cases} x - 2y = 0 \Rightarrow x = 2y \\ z + 3t = 0 \Rightarrow z = -3t \end{cases}$$

$(x,y,z,t) \in W$

$(x,y,z,t) = (2y, y, -3t, t)$

$= (2y, y, 0, 0) + (0, 0, -3t, t)$

$= y(2,1,0,0) + t(0,0,-3,1)$

Base $W = \{(2,1,0,0), (0,0,-3,1)\}$

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 \end{pmatrix} \qquad \dim W = 2$$

Base e dimensão

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$B_2 = \{v_1 = (1,-1) \,;\, v_2 = (2,1)\}$

$\begin{cases} \alpha = 3 \\ \alpha = 2 \end{cases}$ absurd

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i) $\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \therefore S$ é L.I

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$\forall (x,y) \in \mathbb{R}^2$, existem nºs reais α_1, α_2 tais que $(x,y) = \alpha_1(1,-1) + \alpha_2(2,1)$

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$\alpha_1 = x - 2\alpha_2$

$= x - 2\left(\dfrac{x+y}{3}\right) = \dfrac{1}{3}x - \dfrac{2}{3}y$

$\therefore \mathbb{R}^2 = [S]$

Teorema:

Sejam: V e.v. de dimensão n. U e W subespaços vetoriais de V. Então,

$$\boxed{\dim(U+W) = \dim U + \dim W - \dim(U \cap W)}$$

Teorema do Completamento

Se $\dim V = n$ e $\{v_1, \dots, v_k\}$ $(k<n)$ é um conjunto L.I de vetores, então existem $(n-k)$ vetores $u_1, u_2, \dots, u_{n-k}$ de modo que o conjunto $\{v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_{n-k}\}$ é base de V

Obs.: Se $\dim V = n$ e W é s.e de V com dimensão n, então $W = V$

Ex.: 1) $V = \mathbb{R}^2$

$\{(1,-1)\}$ é l.i

$\{(1,-1); (0,1)\}$

Ex 2: $M_2(\mathbb{R})$

$$\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \qquad \begin{matrix} k=3 \\ n=4 \end{matrix} \qquad n-k = 1$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Exercício

$V = \mathbb{R}^4$

114-) $V = \mathbb{R}^4$ $\quad U = \{(a,b,c,d) \mid a+c-d=0\}$

$W = \{(a,b,c,d) \mid a+d=0;\ c-2b=0\}$

Determine uma base e a dimensão de W

" " " " " de $U \cap W$

$\forall u = (x, y, z, t) \in U$ $\qquad a + c - d = 0$

$\qquad\qquad\qquad\qquad\qquad d = a + c$

$u = (x, y, z, t) = (a, b, c, a+c)$

$\qquad = a\underbrace{(1,0,0,1)}_{e_1} + b\underbrace{(0,1,0,0)}_{e_2} + c\underbrace{(0,0,1,1)}_{e_3}$

i) $U = [(1,0,0,1);\ (0,1,0,0);\ (0,0,1,1)] \therefore U = [e_1, e_2, e_3]$

ii) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \{e_1, e_2, e_3\}$ é LI

Logo, dim $U = 3$

b) $\forall w = (x, y, z, t) \in W$ $\qquad a + d = 0 \quad a = -d$

$\qquad\qquad\qquad\qquad\qquad c - 2b = 0 \quad c = 2b$

$w = (x, y, z, t) = (-d, b, 2b, d)$

$\qquad = d\underbrace{(-1,0,0,1)}_{f_1} + b\underbrace{(0,1,2,0)}_{f_2}$

i) $[f_1, f_2] = W$

ii) $\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \Rightarrow \{f_1, f_2\}$ é LI

Logo, dim $W = 2$

c) base de $U + W$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$B_3 = (g_1 = (1,0,0,1); g_2 = (0,1,0,0); g_3 = (0,0,1,1); g_4 = (0,0,0,-2))$

é uma base de $U + W$

Logo, $\dim(U+W) = 4$ Assim, $U+W = \mathbb{R}^4$

d) $\dim(U \cap W) = ?$

$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$4 = 3 + 2 - \dim(U \cap W) \therefore \dim(U \cap W) = 1$

base de $U \cap W$:

$(a,b,c,d) \in U \cap W \Rightarrow \begin{cases} d = a + c & \text{e} \\ a = -d \ ; \ c = 2b \end{cases}$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & -2 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 \end{pmatrix} \sim \begin{cases} a + c + d = 0 \\ -2b + c = 0 \\ -c - 2d = 0 \end{cases}$$

$c = -2d$ $\quad (d, d, -2d, d) = d(-1, 1, -2, 1), d \in \mathbb{R}$

$b = d$ $\quad \therefore B_4 = ((-1, 1, 2, 1))$ é uma base de $U \cap W$

$a = d$

Exercício

1-) Determine o valor (ou valores) de $t \in \mathbb{R}$ para que o conjunto $\{(1,t,1), (0,1,t), (t,1,0)\}$ seja um base do $\mathbb{R}^3$

2-) Dar uma base e a dimensão para o seguinte subespaço do $\mathbb{R}^4$
$S = \{(x,y,z,t) \mid x - z + t = 0 \,;\, x - 3z = 0\}$

3-) Sendo U e W subespaços do $\mathbb{R}^4$ de dimensão 3, determinar a $\dim(U+W)$ sabendo que $U \cap W = [(1,2,1,0), (-1,1,0,1), (1,5,2,1)]$

4-) $V = \mathbb{R}^4$
Dado o subespaço do $\mathbb{R}^4$ $W = [(1,0,-1,2), (0,1,3,-1), (2,-1,-5,5)]$
Determinar:
a) Base e dim W
b) Eq(s) Homongeneas de W
c) Completar para se obter uma base do $\mathbb{R}^4$

1-)
$$\begin{bmatrix} 1 & t & 1 \\ 0 & 1 & t \\ t & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & t & 1 \\ 0 & 1 & t \\ 0 & 1-t^2 & -t \end{bmatrix} \rightarrow \begin{bmatrix} 1 & t & 1 \\ 0 & 1 & t \\ 0 & 0 & -2t+t^3 \end{bmatrix}$$

$\therefore\ -2t + t^3 = 0$ $\qquad t \neq 0$ ou $t \neq \pm\sqrt{2}$

$t(-2+t^2) = 0$

$t = \pm\sqrt{2}$

R: $\boxed{\forall t \in \mathbb{R} - \{0, \sqrt{2}, -\sqrt{2}\}}$

/ /

(2-) $S = \{(x,y,z,t) \mid x - z + t = 0 \; ; \; x - 3z = 0\}$

$x = 3z \qquad t = z - 3z = -2z$

$\forall s = (x,y,z,t) \in S$

$\therefore (x,y,z,t) = (3z, y, z, -2z) = z(3,0,1,-2) + y(0,1,0,0)$

$\therefore S = \{(3,0,1,-2) \; ; (0,1,0,0)\} \qquad \dim S = 2$

(3-) $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$

$\dim(U+V) = 3 + 3 - 2$

$\therefore \dim(U+V) = 4$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 5 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Base $(\cap U W) = \{(1,2,1,0)(0,3,1,1)\}$

$\dim = 2$

(4-) $a(1,0,-1,2) + b(0,1,3,-1) + c(2,-1,-5,5) = (x,y,z,t)$

$$\begin{cases} a + 0 + 2c = x \\ 0 + b - c = y \\ -a + 3b - 5c = z \\ 2a - b + 5c = t \end{cases} \sim \begin{pmatrix} 1 & 0 & 2 & x \\ 0 & 1 & -1 & y \\ -1 & 3 & -5 & z \\ 2 & -1 & 5 & t \end{pmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & x \\ 0 & 1 & -1 & y \\ 0 & 3 & -3 & x+z \\ 0 & -1 & 1 & t-2x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & x \\ 0 & 1 & -1 & y \\ 0 & 0 & 0 & x+z-3y \\ 0 & 0 & 0 & -2x+y+t \end{bmatrix}$$

Eq. homogênea $= \begin{cases} x + z - 3y = 0 \text{ e} \\ -2x + y + t = 0 \end{cases}$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 2 & -1 & -5 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & -1 & -3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Base $W = \{(1, 0, -1, 2), (0, 1, 3, -1)\}$ $\dim = 2$

c) Base $W \in \mathbb{R}^4 = \{(1,0,-1,2), (0,1,3,-1), (0,0,1,1), (0,0,0,1)\}$

Exercícios

1-) Sendo A e B matrizes inversíveis de mesma ordem, exprimir a matriz X, onde

$$X^t(A^t - I)^{-1} = B$$

2-) Para que valor(es) x, y, z a matriz

$$M = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ x & y & z \end{bmatrix}$$ multiplicada pela sua transposta é igual a matriz identidade.

3-) Seja $V = P_4(\mathbb{R})$ (def de dependência linear) Para que valor de m os vetores:

$p_1 = 3 + mx + 12x^2 - 4x^3 + x^4$

$p_2 = 1 + x^2$

$p_3 = x + 3x^2 - x^3 + x^4$

$p_4 = x^2 + x^4$; são LD

4-) Dados os subespaço de $V = \mathbb{R}^4$: $U = \{(x, y, z, t) \mid x - y = z + t ; x - y - 2t = 0\}$ e $W = [(1,0,1,1); (0,1,-1,2)]$, determinar:

a-) Uma base de $U \cap W$

b-) " " " $U + W$

5-) Seja $V = M_2(\mathbb{R})$

Dado o subconjunto de $M_2(\mathbb{R})$ $S = \left\{ \begin{pmatrix} x-y & y \\ -y & x+y \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$

a) S é s.e de V? Justifique

b) Dar uma base e a dimensão de S.

/ /

1.) $X^t(A^t - I)^{-1} = B$

$X^t = B(A^t - I)$ $\qquad X = B^t(I^t - A)$

$\cancel{X = I^t A +}$ $\qquad X = B^t(I - A)$

2.) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ x & y & z \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & x \\ 1/\sqrt{2} & 0 & y \\ 0 & 1 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 0 \qquad x = \frac{-y}{\sqrt{2}} \cdot \sqrt{2} \quad \therefore \quad x = -y \qquad y = -x$

$z = 0 \qquad\qquad x^2 + x^2 + 0^2 = 1$

$x^2 + y^2 + z^2 = 1 \qquad 2x^2 = 1$

$x = \pm\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} \cdot \sqrt{2} = \frac{1}{2} + \frac{y}{\sqrt{2}} = 0$

$x = \frac{1}{\sqrt{2}}\,;\ y = \frac{-\sqrt{2}}{2}\,;\ z = 0$

$\frac{\sqrt{2} + 2y}{2\sqrt{2}} = 0 \quad \therefore \quad y = \frac{-\sqrt{2}}{2}$

ou

$x = \frac{-1}{\sqrt{2}}\,;\ y = \frac{-\sqrt{2}}{2}\,;\ z = 0$

3.) $a(3 + mx + 12x^2 - 4x^3 + x^4) + b(1 + x^2) + c(x + 3x^2 - x^3 - x^4) + c$

$d(x^2 + x^4) = 0x + 0x^2 + 0x^3 + 0x^4$

$$\begin{cases} 3a + b + c = 0 \\ am + 0 + c = 0 \\ 12a + b + 3c + d = 0 \\ -4a + 0 - c + 0 = 0 \\ a + 0 - c + d = 0 \end{cases}$$

$am + c = 0 \qquad c = -4a$

$m = \frac{-c}{a} = -\frac{(-4a)}{a} = 4$

$\therefore m = 4$

$x - y = z + t = 0 \quad x - y = 0$
$z + t = 0$

$\begin{cases} x-y-z-t=0 \\ x-y-2t=0 \end{cases}$

$x - y - z - t$

1-) $U = \{(x,y,z,t) \mid x-y = z+t,\ x-y-2t=0\}$
$W = [(1,0,1,1),(0,1,-1,2)]$

$\begin{cases} x-y-z-t=0 \\ x-y+0-2t=0 \end{cases} \sim \begin{cases} x-y-z-t=0 \\ 0\ \ 0\ -z+t=0 \end{cases}$

$t = z \qquad x = y + z + t$
$\qquad\qquad x = y + t + t \quad \therefore x = y + 2t$

$\forall u = (x,y,z,t) \in U$:
$(x,y,z,t) = (y+2z, y, z, z) = y(1,1,0,0) + z(2,0,1,1)$

$U = \{(1,1,0,0),(2,0,1,1)\}$

— " —

$W = \{(1,0,1,1);(0,1,-1,2)\}$

Estudar

Não esquecer de escalonar

b) $U + W = [(1,1,0,0);(2,0,1,1),(1,0,1,1);(0,1,-1,2)]$

Cont. no verso

a) $U \cap W \Rightarrow$

$\begin{cases} x-y-z-t=0 \\ x-y+0-2t=0 \\ x+0+z+t=0 \\ 0+y-z+2t=0 \end{cases} \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & -2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

(Não possui $U \cap W$; pois não é LI e não forma base).

$\begin{cases} x-y-t-z=0 \\ y+2t+2z=0 \\ t-z=0 \\ -3t=0 \end{cases}$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad LI$$

$\therefore \mathcal{U} + W = \{(1,1,0,0);(0,-1,1,1),(0,0,1,1);(0,0,0,3)\}$

5-)

$$S = \left\{ \begin{pmatrix} x-y & y \\ -y & x+y \end{pmatrix} \middle| x,y \in \mathbb{R} \right\}$$

i) $x=0$ e $y=0$

$$\begin{pmatrix} 0-0 & 0 \\ -0 & 0+0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad V$$

ii) $$\underbrace{\overset{\in S}{\begin{pmatrix} x_1-y_1 & y_1 \\ -y_1 & x_1+y_1 \end{pmatrix}}}_{A} + \underbrace{\overset{\in S}{\begin{pmatrix} x_2-y_2 & y_2 \\ -y_2 & x_2+y_2 \end{pmatrix}}}_{B} = \underbrace{\begin{pmatrix} \overset{x-}{x_1-y_1+x_2-y_2} & \overset{y}{y_1+y_2} \\ \overset{-(y_1+y_2)}{-y_1-y_2} & x_1+y_1+x_2+y_2 \end{pmatrix}}_{A+B \in S} \quad V$$

iii) $$\lambda \begin{pmatrix} x-y & y \\ -y & x+y \end{pmatrix} = \begin{pmatrix} \lambda(x-y) & \lambda y \\ \underset{-(\lambda y)}{\lambda(-y)} & \lambda(x+y) \end{pmatrix} \quad V$$

b) $S = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \right]$

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\alpha - \beta = 0 \qquad -\beta = 0 \qquad \alpha = \beta \qquad \therefore \alpha = \beta = 0 \qquad \therefore LI$

$\beta = 0 \qquad \alpha + \beta = 0 \qquad \alpha = 0$

$W = [(1,0,1,1);(0,1,-1,2)]$

$w \in \mathbb{R}^4 \qquad w = a(1,0,1,1) + b(0,1,-1,2)$

$\Downarrow$

$(x,y,z,t) = a(1,0,1,1) + b(0,1,-1,2)$

$$\begin{cases} a = x \\ b = y \\ a - b = z \\ a + 2b = t \end{cases}$$

$a = x$ e $b = y$

$\Rightarrow a - b = z \Rightarrow x - y = z \therefore x - y - z = 0$

$\Rightarrow a + 2b = t \Rightarrow x + 2y = t \therefore x + 2y - t = 0$

$\therefore$ Equações homogenicas:

$$\begin{cases} x - y - z = 0 \\ x + 2y - t = 0 \end{cases} \sim \begin{cases} x - y - z = 0 \\ 0 + 3y + z - t = 0 \end{cases}$$

$$\begin{cases} x - y - z = 0 \\ 3y + z - t = 0 \\ x - y - z - t = 0 \\ x - y - 2t = 0 \end{cases} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\begin{cases} x - y - z = 0 \\ 3y + z - t = 0 \\ z - 2t = 0 \\ -t = 0 \end{cases}$$

$\therefore U \cap W = \{(0,0,0,0)\}$ ~~não é base~~

$\dim U \cap W = 0$

$\nexists$ base

Resumo teórico de Álgebra

- Matrizes

-> Matriz transposta

- $(A+B)^t = A^t + B^t$
- $(\lambda A)^t = \lambda A^t$
- $(A^t)^t = A$
- $(AB)^t = B^t A^t$

* Uma matriz quadrada é simétrica quando $A = A^t$
* Uma matriz quadrada é antisimétrica quando $A = -A^t$

Soma de matrizes simétricas é simétrica?

Prova: $A = A^t$ e $B = B^t$

$(A+B)^t = A^t + B^t = A + B \therefore A+B$ é simétrica

Soma de matrizes antisimétrica

Prova: $A = -A^t \quad B = -B^t$

$(A+B)^t = A^t + B^t = -A - B = -(A+B) \therefore A+B$ é anti-simétrica

Matriz quadrada pode ser escrita por duas matrizes utilizando a seguinte identidade:

$$A = \frac{A + A^t}{2} + \frac{A - A^t}{2} \qquad B = \frac{A + A^t}{2} \qquad C = \frac{A - A^t}{2}$$

Produto de matrizes simétricas

$(AB)^t = B^t A^t = BA \quad \therefore AB \neq BA$, com isso nem sempre o produto é simétrico

-> Matriz quadrada Inversível

$AB = BA = I_n \Rightarrow B = A^{-1}$

- $(A^{-1})^{-1} = A$
- $(\lambda A)^{-1} = \lambda A^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$

* Para que uma matriz seja inversível, seu determinante tem que ser diferente de $\emptyset$

-> Cálculo da matriz inversa:

· Pode ser pelo seguinte definição: $AB = I_n$,
ou pelo método de Binet

$$A^{-1} = \frac{1}{\det A} (\operatorname{cof} A)^t$$

-> Matriz semelhantes

→ Duas matrizes quadrados são semelhantes, quando elas possui a seguinte condição:

$BP = AP \Rightarrow B = P^{-1}AP$

* Uma matriz é ortogonal quando $A.A^t = I_n \Rightarrow A^t = A^{-1}$
* " " " idempotente se $A^2 = A$

Matrizes

- Uma matriz é inversível, quando seu determinante for diferente de 0

  $AX = I \quad X = A^{-1}$, apenas se $\det(X) \neq 0$

- Uma matriz quadrada é simétrica quando $A = A^t$
- Uma matriz é antisimétrica quando $A = -A^t$
- Uma matriz é ortogonal quando $AA^t = I \rightarrow A^t = A^{-1} \rightarrow \det A = \pm 1$

Propriedade do determinante

$\det A^{-1} = \dfrac{1}{\det A} \qquad \det A = \det A^t \qquad \det(AB) = \det A . \det B$

Discussão de um sistema linear por escalonamento

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} & B_1 \\ 0 & A_{22} & \cdots & A_{2n} & B_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{mn} & B_m \end{bmatrix}$$

Analisando o sistema, com $B_m \neq 0$, temos os seguintes casos

$A_{mn} \neq 0$ $\begin{cases} m = n \Rightarrow \text{Sistema é compatível e determinado (1 solução)} \\ m < n \Rightarrow \text{Sistema é compatível e indeterminado (n solução)} \end{cases}$

$A_{mn} = 0$ O sistema é incompatível, e $B_m \neq 0$

Álgebra Linear e suas aplicações

1-)

$$A = \begin{bmatrix} 2 & -1 & -2 \\ 1 & 4 & -1 \\ 2 & 1 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 5 & 7 & 9 \\ 3 & 2 & 1 & 0 \\ 5 & 7 & 9 & 11 \\ 7 & 6 & 5 & 4 \end{bmatrix}$$

2-)

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 4 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \\ 2 & 3 \\ 4 & 2 \end{bmatrix} \text{ e } D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

a-) $A + 2B - C$

$$\begin{bmatrix} 1 & 9 \\ 1 & 8 \\ -2 & 6 \end{bmatrix}$$

b-) $\dfrac{A+2C}{2} + \dfrac{B}{3} \Rightarrow \dfrac{3A+6C+2B}{6} \sim \dfrac{3A+2B+6C}{6}$

$$\begin{bmatrix} 1/2 & 1 \\ 13/6 & 35/6 \\ 17/3 & 19/3 \end{bmatrix}$$

c-) $2A - \dfrac{4}{3}B - \dfrac{8}{3}C \sim \dfrac{6A-4B-8C}{3}$

$$\begin{bmatrix} 2 & 8/3 \\ -10 & -22/3 \\ -4/3 & 0 \end{bmatrix}$$

d) $AD \Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 2 & 5 \\ 4 & 8 & 12 \end{bmatrix}$

$\underbrace{3 \times 2 = 2 \times 3}_{3 \times 3}$

e) $DA \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 3 & 11 \end{bmatrix}$

$\underbrace{2 \times 3 = 3 \times 2}_{2 \times 2}$

f) $(B+C)\,D$ $\quad B+C = \begin{bmatrix} 0 & 2 \\ 4 & 7 \\ 3 & 4 \end{bmatrix}$ $\quad (B+C).D \Rightarrow \begin{bmatrix} 0 & 2 \\ 4 & 7 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} =$

$\underbrace{3 \times 2 = 2 \times 3}_{3 \times 3}$

$= \begin{bmatrix} 0 & 2 & 4 \\ 4 & 11 & 18 \\ 3 & 7 & 11 \end{bmatrix}$

g) $(CD)^t$ $\quad CD = \begin{bmatrix} 0 & -1 \\ 2 & 3 \\ 4 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 2 & 5 & 8 \\ 4 & 6 & 8 \end{bmatrix}$

$\underbrace{3 \times 2 = 2 \times 3}_{3 \times 3}$

$(CD)^T = \begin{bmatrix} 0 & 2 & 4 \\ -1 & 5 & 6 \\ -2 & 8 & 8 \end{bmatrix}$

3-) $A = mB + nC$

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 4 \end{bmatrix} = m \begin{bmatrix} 0 & 3 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} + n \begin{bmatrix} 0 & -1 \\ 2 & 3 \\ 4 & 2 \end{bmatrix}$$

$1 = 0m + 0n$ $\quad \nexists$ combinação Linear

4) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \Rightarrow \Delta = 0$ $\quad \therefore$ Não existe Matriz inversa

5) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad C = \begin{bmatrix} 3 & 4 \\ 2 & 0 \end{bmatrix}$

a-) $AX = B$ $\quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$

$\begin{cases} a + b = 2 \\ b = 2 \end{cases}$ $\quad a = 0$, $b = 2$

$\begin{cases} c + d = 0 \\ d = 1 \end{cases}$ $\quad c = -1$, $d = 1$

$X = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$

b-) $X = (A^t + 2B)(B - C^t)$.

$A^t = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad 2B = \begin{bmatrix} 4 & 0 \\ 4 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad C^T = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}$

$$X = \begin{bmatrix} 5 & 0 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -10 \\ -11 & -7 \end{bmatrix}$$

c) $B + AX^t = 2C \rightarrow AX^t = 2C - B$

$$AX^t = \begin{bmatrix} 4 & 8 \\ 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X^t = \begin{bmatrix} 4 & 8 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a+b & c+d \\ b & d \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 2 & -1 \end{bmatrix}$$

$a+b = 4 \therefore a = 2 \qquad c+d = 8 \therefore c = 9$

$b = 2 \qquad d = -1$

$$X^t = \begin{bmatrix} 2 & 9 \\ 2 & -1 \end{bmatrix} \therefore X = \begin{bmatrix} 2 & 2 \\ 9 & -1 \end{bmatrix}$$

d) $(XA)^t = B \qquad X^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$X^t A^t = B$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} a+c & c \\ b+d & d \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

$a + c = 2 \qquad a = 2 - c = 2 - 0 = 2$

$b + d = 1 \qquad b = 1 - d = 1 - 1 = 0$

$c = 0$

$d = 1$

$$X^t = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \therefore X = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

6-) $A = \begin{bmatrix} 1 & 2 \\ m & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 2 & m \end{bmatrix}$ $A+B = B+A \Rightarrow A+B = \begin{bmatrix} 0 & 2 \\ m+2 & m+1 \end{bmatrix}$ $B+A = \begin{bmatrix} 0 & 2 \\ 2+m & 1+m \end{bmatrix}$

$\therefore A+B = B+A$

$AB = BA \Rightarrow AB = \begin{bmatrix} 4 & 2m \\ -m+2 & m \end{bmatrix}$ $BA = \begin{bmatrix} -1 & -2 \\ 2+m^2 & 4+m \end{bmatrix}$ $\therefore AB \neq BA$

$\therefore$ A matriz A e B não se comutem

7-) $H = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$ (a_{ij})

8-) $H = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$

9-) $A^2 = A$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\therefore$ Não é idempotente

10-) $\frac{1}{3} \cdot \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \cdot \frac{1}{3} \cdot \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$\therefore$ Esta matriz é idempotente

11-) $A = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ $A^T = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$ $A^{-1} = \frac{1}{\det A} (\operatorname{cof} A)^t$

$\det A = 5/2$ $\frac{1}{\det A} = 2/5$ $\operatorname{cof} A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$ $(\operatorname{cof} A)^t = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \cdot \frac{2}{5}$

12-) Determinar a,b,c para que a matriz M seja ortogonal

$$M=\begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & -2/3 & b \\ 2/3 & 1/3 & c \end{bmatrix} \times \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$1/9 + 4/9 + a^2 = 1 \Rightarrow 5/9 + a^2 = 1 \Rightarrow a^2 = 1 - 5/9 = 4/9 \Rightarrow a = 2/3$

$4/9 + 4/9 + b^2 = 1 \Rightarrow 8/9 + b^2 = 1 \Rightarrow b^2 = 1 - 8/9 = 1/9 \Rightarrow b = 1/3$

$4/9 + 1/9 + c^2 = 1 \Rightarrow 5/9 + c^2 = 1 \Rightarrow c^2 = 1 - 5/9 = 4/9 \Rightarrow c = 2/3$

R: $(a,b,c) = (2/3;\ 1/3;\ 2/3)$ ou $(-2/3, -1/3, -2/3)$

13-) $A=\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & 2 & -5 \end{bmatrix}$ $A^K = 0;\ K \in \mathbb{Z}^+$

?

14-) $AX=B$ $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

$a + b = 2$ $a = 2 - 3 = -1$ $X = \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix}$

$c + d = 1$ $c = 1 - 0 = 1$

$b = 3$

$d = 0$

/ /

15-) $A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$

$\det A = 1 - abc + abc + b^2 + c^2 + a^2$

$\det A = a^2 + b^2 + c^2 + 1 \neq 0$

16-) $D_0 = \begin{bmatrix} -sen\theta & -\cos\theta \\ \cos\theta & -sen\theta \end{bmatrix}$ $\underline{AB = I_n}$ $A^{-1} = \frac{1}{\det A} (\operatorname{cof} A)^t$ $\operatorname{cof} A = \begin{bmatrix} -sen\theta & -\cos\theta \\ +\cos\theta & -sen\theta \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -sen\theta & \cos\theta \\ -\cos\theta & -sen\theta \end{bmatrix}$

$\begin{bmatrix} -sen\theta & -\cos\theta \\ \cos\theta & -sen\theta \end{bmatrix} \cdot \begin{bmatrix} -sen\theta & \cos\theta \\ -\cos\theta & -sen\theta \end{bmatrix} =$

$\Rightarrow = \begin{bmatrix} sen^2\theta + \cos^2\theta & -sen\theta\cos\theta + sen\theta\cos\theta \\ -sen\theta\cos\theta + sen\theta\cos\theta & \cos^2\theta + sen^2\theta \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

17-) $A = \begin{bmatrix} x^2 & 0 \\ 2 & y+z \end{bmatrix}$ $B = \begin{bmatrix} 4 & z \\ y & -x \end{bmatrix}$ $A = B^t$

$\begin{bmatrix} x^2 & 0 \\ 2 & y+z \end{bmatrix} = \begin{bmatrix} 4 & y \\ z & -x \end{bmatrix}$ $x^2 = \cdot 4 \therefore x = \cdot 2$; $y = 0$; $z = 2$

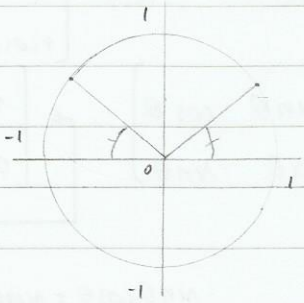
$y + z + x = 0 \Rightarrow x = -y - z = -0 - 2 = -2$

$\begin{vmatrix} -2 & 0 & -1 \\ +2 & 1 & 1 \\ 4 & 5 & 2 \end{vmatrix} = -4 - 10 + 4 + 10 = 0$

18-) $\begin{bmatrix} t^2 - 1 & t \\ 1 & 2t \end{bmatrix}$ Nunca será matriz nula

Mas e as outras condições

19-) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & \operatorname{sen} x & 0 \\ 0 & 2 & \cos x \end{bmatrix} \Rightarrow \begin{vmatrix} 1 & 0 & 2 \\ 2 & \operatorname{sen} x & 0 \\ 0 & 2 & \cos x \end{vmatrix} = \operatorname{sen} x \cos x + 8$



$\operatorname{sen} 45 \cos 45 + 8$

$\frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} + 8 = \pm \frac{2}{4} + 8 = \pm \frac{1}{2} + 8 = \frac{17}{2}$ ou $\frac{15}{2}$

$\therefore \frac{15}{2} \leq \det A \leq \frac{17}{2}$

20-) $\det \begin{bmatrix} x & 0 & 1 \\ 1 & x & 0 \\ 0 & 1 & x \end{bmatrix} = 0 \quad \begin{vmatrix} x & 0 & 1 \\ 1 & x & 0 \\ 0 & 1 & x \end{vmatrix} = x^3 + 1 = 0 \quad x^3 = -1 \therefore \underline{x = -1}$

21-) $\det(A^{-1}) = \frac{1}{\det A} \qquad AA^{-1} = I_n$

$\det(A^{-1}) \det A = 1$

$\det(AA^{-1}) = \det(A^{-1}) \det A = \det(AA^{-1}) = \det(I_n) = 1$

Com isso, provamos que $\det(A^{-1}) = \frac{1}{\det A}$

22-)

?

Confuso

24-) ~~23-)~~

$$AD = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} = \begin{bmatrix} a_{11}d_{11} & a_{12}d_{22} & a_{13}d_{33} \\ a_{21}d_{11} & a_{22}d_{22} & a_{23}d_{33} \\ a_{31}d_{11} & a_{32}d_{22} & a_{33}d_{33} \end{bmatrix}$$

$$DA = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}d_{11} & a_{21}d_{11} & a_{31}d_{11} \\ a_{21}d_{22} & a_{22}d_{22} & a_{32}d_{22} \\ a_{11}d_{33} & a_{23}d_{33} & a_{33}d_{33} \end{bmatrix}$$

~~24-)~~ 23- Como $AC = CA \therefore C = A^{-1}CA$ e

$BC = CB \therefore C = CBB^{-1}$

$$A^{-1}CA = CBB^{-1}$$

$$A^{-1}ACC^{-1} = BB^{-1}$$

$$AB = BA$$

25-) $A.\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.A$

$$\begin{bmatrix} m & n \\ s & t \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} m & n \\ s & t \end{bmatrix}$$

$$\begin{bmatrix} am & bn \\ as & bt \end{bmatrix} = \begin{bmatrix} am & an \\ bs & bt \end{bmatrix} \Rightarrow \begin{matrix} am = am \\ bt = bt \end{matrix}$$

$as = bs \Rightarrow as - bs = 0 \Rightarrow s(a-b) = 0$

$\therefore s = 0$

$bn = an \Rightarrow bn - an \Rightarrow n(b-a) = 0$

$\therefore n = 0$

$$A = \begin{bmatrix} m & 0 \\ 0 & t \end{bmatrix}; \forall m, t \in \mathbb{R}$$

26-) $Tr(A) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \dots + a_{nn}$; Mostre $Tr(AB) = Tr(BA)$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ $\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$

$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{11} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

$BA = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{21} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$

$Tr(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}$

$Tr(BA) = b_{11}a_{11} + b_{12}a_{21} + b_{21}a_{12} + b_{22}a_{22}$

$Tr(AB) = Tr(BA)$

$\underline{a_{11}b_{11}} + \underline{a_{12}b_{21}} + \underline{a_{21}b_{12}} + \underline{a_{22}b_{22}} = \underline{b_{11}a_{11}} + \underline{b_{12}a_{21}} + \underline{b_{21}a_{12}} + \underline{b_{22}a_{22}}$

27-) A) $(XM)^t = N$

$M^t X^t = N$

$(M^t X^t)^t = N^t$

$XM = N^t$

$\underline{X = N^t M^{-1}}$

B) $(XM)^{-1} = N$

$XM = N^{-1}$

$\underline{X = N^{-1} M^{-1}}$

C) $M(XM)^{-1} = N$

$(XM)^{-1} = NM^{-1}$

$M^{-1}X^{-1} = NM^{-1}$

$X^{-1} = N$

$\underline{X = N^{-1}}$

/ /

D) $M(XM^{-1})^t = N$

$(XM^{-1})^t = NM^{-1}$

$(M^{-1})^t X^t = NM^{-1}$

$X^t = NM^{-1}M^t$

$X = (NM^{-1}M^t)^t$

$X = N^t(M^{-1})^t M$

$X = (M^{-1}N)^t M$

E) $M[(XN)^t]^{-1} = N$

$[(XN)^t]^{-1} = NM^{-1}$

$(XN)^t = (NM^{-1})^{-1}$

$(XN)^t = MN^{-1}$

$XN = (MN^{-1})^t$

$XN = (N^{-1})^t M^t$

$X = (N^{-1})^t M^t N^{-1}$

28-) Considere $B = P^{-1}AP$

I) B^t é inversível e $(B^t)^{-1} = (B^{-1})^t$ I-) Verdadeira

$d(B^t) \neq 0$

$\det(B) = \det(P^{-1}AP) \Rightarrow \det(B) = \cancel{\det}(P^{-1}) \det(A) \cancel{\det}(P)$

$\therefore \det(B) = \det A \neq 0 \quad \therefore \quad B^t$ é inversível

$BB^{-1} = In \Rightarrow (BB^{-1})^t = I^t \Rightarrow (B^{-1})^t B^t = I^t \Rightarrow (B^{-1})^t = (B^t)^{-1}$

II - A e B é simétrica $B = P^{-1}AP$ II-) Falsa

$A = A^t \quad B = B^t$ $B^t = (P^{-1}AP)^t = B^t = (AP)^t(P^{-1})^t \Rightarrow$

$B^t = P^t A^t (P^{-1})^t$

Só é verdadeira se $P^t = (P^{-1})^t \Rightarrow P^t = P^{-1}$, ou seja se P é ortogonal

spiral

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III-) $\det(A-\lambda I) = \det(B-\lambda I)$

$B = P^{-1}AP$

$\det(P^{-1}AP - \lambda I) = \det P^{-1} \det A$?

29-) $A = \begin{pmatrix} 3 & a_{12} & 7 \\ 10 & a_{22} & a_{23} \\ 5 & a_{32} & a_{33} \end{pmatrix}$ $\Sigma L = \Sigma c = \Sigma d = S$

$S = 3 + 10 + 5 = 18$

$a_{12} = 18 - 7 - 3 = 8$ $a_{23} = 18 - 10 - 6 = 2$

$a_{22} = 18 - 7 - 5 = 6$ $a_{33} = 18 - 5 - 4 = 9$

$a_{32} = 18 - 8 - 6 = 4$

$\therefore A = \begin{bmatrix} 3 & 8 & 7 \\ 10 & 6 & 2 \\ 5 & 4 & 9 \end{bmatrix}$ $\frac{A}{2} = \begin{pmatrix} 3/2 & 4 & 7/2 \\ 5 & 3 & 1 \\ 5/2 & 2 & 9/2 \end{pmatrix}$

$\det\left(\frac{A}{2}\right) = \frac{81}{4} + 10 + 35 - \frac{105}{4} - 3 - 90 = -54$

$\det A = \underline{-432}$

Sistemas Lineares

30-) $\begin{cases} x + 2y + z = 4 \\ 3x - 2y + z = 2 \\ 4x + 3y - 2z = 5 \end{cases}$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & -2 & 1 & 2 \\ 4 & 3 & -2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -8 & -2 & -10 \\ 0 & -5 & -6 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -8 & -2 & -10 \\ 0 & 0 & 38 & 38 \end{bmatrix}$$

$\begin{cases} x + 2y + z = 4 \\ -8y - 2z = -10 \\ 38z = 38 \end{cases}$ $z = \frac{38}{38} = 1$ $y = \frac{-10 + 2z}{-8} = \frac{-10 + 2}{-8} = 1$

$z = 4 - x - 2y = 4 - 1 - 2 = 1$

R: $(1, 1, 1)$ Sistema compatível e determinado

31-) $\begin{cases} x + 2y + z = 4 \\ 3x - 2y + z = 2 \\ 5x + 2y + 3z = 10 \end{cases}$ Sistema compatível e indeterminado

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & -2 & 1 & 2 \\ 5 & 2 & 3 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -8 & -2 & -10 \\ 0 & -8 & -2 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -8 & -2 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} x + 2y + z = 4 \\ -8y - 2z = -10 \end{cases}$ $y = \frac{-10 + 2z}{-8} = \frac{-5 + z}{-4} = \frac{5 - z}{4}$

$x = 4 - 2y - z$

$\Rightarrow 4 - 2 \cdot \frac{1}{4}(5 - z) - z$

$\Rightarrow 4 - \frac{1}{2}(5 - z) - z$ $\rightarrow 4 - \frac{5}{2} + \frac{z}{2} - z = \frac{3 - z}{2}$

$(x, y, z) = \left(\frac{3 - z}{2}; \frac{5 - z}{4}; z\right)$

spiral

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32-) $\begin{cases} x + 2y + z = 4 \\ 3x - 2y + z = 2 \\ 2x - 4y = 5 \end{cases}$ Sistema Impossível (Incompatível)

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & -2 & 1 & 2 \\ 2 & -4 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -8 & -2 & -10 \\ 0 & -8 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -8 & -2 & -10 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

33-) $\begin{cases} x + 2y + z + t = 3 \\ 2x - y + 3t = 1 \\ 3x + y - z + 2t = 4 \end{cases}$ Sistema compatível e indeterminado

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & -1 & 0 & 3 & 1 \\ 3 & 1 & -1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 3 \\ 0 & -5 & -2 & 1 & -5 \\ 0 & -5 & -4 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 3 \\ 0 & -5 & -2 & 1 & -5 \\ 0 & 0 & -2 & -2 & 0 \end{bmatrix}$$

$\begin{cases} x + 2y + z + t = 3 \\ -5y - 2z + t = -5 \\ -2z - 2t = 0 \end{cases}$ $z = \dfrac{2t}{-2} = -t$

$-5y = -5 + 2z - t$

$-5y = -5 - 2t - t$

$y = \dfrac{-5 - 3t}{-5} = \dfrac{5 + 3t}{5}$

$(x, y, z, t) = \left(\dfrac{5 - 3t}{5} ; \dfrac{5 + 3t}{5} ; -t ; t \right) t \in \mathbb{R}$

$x = 3 - 2y - z - t$

$x = 3 - 2\left(\dfrac{5 + 3t}{5}\right) + t - t$

$x = \dfrac{15 - 10 - 3t}{5} = \dfrac{5 - 3t}{5}$

34-) $\begin{cases} x + 2y + z + t = 3 \\ 2x - y + 3t = 1 \\ x - 3y + 2z - t = 2 \end{cases}$ Sistema compatível e indeterminado

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & -1 & 0 & 3 & 1 \\ 1 & -3 & 2 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 0 & -5 & -2 & 1 & -5 \\ 0 & -5 & 1 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 0 & -5 & -2 & 1 & -5 \\ 0 & 0 & 3 & -3 & 4 \end{pmatrix}$$

$\begin{cases} x + 2y + z + t = 3 \\ -5y - 2z + t = -5 \\ 3z - 3t = 4 \end{cases}$ $z = \dfrac{4+3t}{3}$ $y = \dfrac{-5 + 2z - t}{-5}$

$$y = \frac{-5 + \frac{2}{3}(4+3t) - t}{-5}$$

$$y = \frac{-15 + 2(4+3t) - 3t}{-15} = \frac{-15 + 8 + 6t - 3t}{-15}$$

$y = \dfrac{-7+3t}{-15} = \dfrac{7-3t}{15}$ $x = 3 - 2y - z - t$

$$x = 3 - \frac{2}{15}(7-3t) - \frac{1}{3}(4+3t) - t$$

$$x = \frac{45 - 14 + 6t - 20 - 15t - 15t}{15} = \frac{11 - 24t}{15}$$

$\therefore\ x = \dfrac{11-24t}{15}$; $y = \dfrac{7-3t}{15}$; $z = \dfrac{4+3t}{3}$; $t \in \mathbb{R}$

35-) $\begin{cases} x + 2y + z + t = 3 \\ 2x - y + 3t = 1 \\ 3x + y - z + 2t = 4 \\ 4x + 3y - 2z + 4t = 7 \end{cases}$ Sistema compatível e determinado

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & -1 & 0 & 3 & 1 \\ 3 & 1 & -1 & 2 & 4 \\ 4 & 3 & -2 & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 0 & -5 & -2 & 1 & -5 \\ 0 & -5 & -4 & -1 & -5 \\ 0 & -5 & -6 & 0 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 0 & -5 & -2 & 1 & -5 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -4 & -1 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 0 & -5 & -2 & 1 & -5 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix} \quad \begin{cases} x + 2y + z + t = 3 \\ -5y - 2z + t = -5 \\ -2z - 2t = 0 \\ 3t = 0 \end{cases}$$

$t = 0$

$z = \frac{2t}{-2} = \frac{2.0}{-2} = 0$

$y = \frac{-5}{-5} = 1$

$x = 3 - 2y = 3 - 2.1 = 1$

$\therefore (x, y, z, t) = (1, 1, 0, 0)$

36-) $\begin{cases} x + y + z + t = 2 \\ 2x - y + 3t = 1 \\ 3x + y - z + 2t = 4 \\ 4x + 3y - 2z + 4t = 8 \end{cases}$ Sistema Compatível e Determinado

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 0 & 3 & 1 \\ 3 & 1 & -1 & 2 & 4 \\ 4 & 3 & -2 & 4 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & -2 & 1 & -3 \\ 0 & -2 & -4 & -1 & -2 \\ 0 & -1 & -6 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & -2 & 1 & -3 \\ 0 & 0 & -8/3 & -5/3 & 0 \\ 0 & 0 & -16/3 & -1/3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & -2 & 1 & -3 \\ 0 & 0 & -8/3 & -5/3 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix} \qquad \begin{cases} x + y + z + t = 2 \\ -3y - 2z + t = -3 \\ -8/3\, z - 5/3\, t = 0 \\ 3t = 1 \end{cases}$$

$t = 1/3$

$$z = \left(\frac{5}{3} \cdot \frac{1}{\cancel{3}} \right) \cdot \frac{\cancel{3}}{(-8)} = \frac{-5}{24}$$

$$y = \frac{-3 - t + 2z}{-3}$$

$$y = \frac{-3 - \frac{1}{3} - \frac{5}{12}}{-3}$$

$$y = \frac{-36 - 4 - 5}{-36} = \frac{-45}{-36} = \frac{5}{4}$$

$$x = 2 - t - z - y = 2 - \frac{1}{3} + \frac{5}{24} - \frac{5}{4} = \frac{48 - 8 + 5 - 30}{24} = \frac{15}{24} = \frac{5}{8}$$

$$(x, y, z, t) = \left(\frac{5}{8}, \frac{5}{4}, \frac{-5}{24}, \frac{1}{3} \right)$$

37.)

38-) $x + y + z =$

$50x + 60y + 100z = 10000$

39-) $\begin{cases} x + 2y - 3z = 0 \\ 3x - 2y + z = 0 \\ mx - 14y + 15z = 0 \end{cases}$

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & -2 & 1 \\ m & -14 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -8 & 10 \\ 0 & -14-2m & 15+3m \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 5 \\ 0 & -14-2m & 15+3m \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 5 \\ 0 & -2m & \frac{-5+6m}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 5 \\ 0 & 0 & \frac{m-5}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 5 \\ 0 & 0 & m-5 \end{bmatrix}$$

O sistema é compatível e determinado se $m \neq 5$

" " " indeterminado se $m = 5$

40-) $\begin{cases} 3x - 2y + z = b \\ 5x - 8y + 9z = 3 \\ 2x + y + az = -1 \end{cases}$; $a, b \in \mathbb{R}$

$$\begin{bmatrix} 3 & -2 & 1 & b \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & -1 \end{bmatrix} \xrightarrow[3L_3 - 2L_1]{3L_2 - 5L_1} \begin{bmatrix} 3 & -2 & 3 & b \\ 0 & -14 & 22 & 9-5b \\ 0 & 7 & 3a-2 & -3-2b \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 3 & -2 & 3 & b \\ 0 & -14 & 22 & 9-5b \\ 0 & 0 & 6a+18 & 3-9b \end{bmatrix}$$

-> Se $a \neq \frac{-18}{6} \neq -3$; o Sistema é compatível e determinado

-> Se $a = -3$ e $b = \frac{1}{3}$; o Sistema é compatível e indeterminado

-> Se $a = -3$ e $b \neq \frac{1}{3}$; o Sistema é incompatível

41-) Se $\{x, y, z\}$ é $\begin{cases} x + 2y + z = 1 \\ y + 2z = 4 \\ x + y + z = 2 \end{cases}$, então $\frac{xy}{z} = ?$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

$\begin{cases} x + 2y + z = 1 \\ y + 2z = 4 \\ 2z = 5 \end{cases}$ $z = \frac{5}{2}$ $y = 4 - 2z = 4 - 2 \cdot \frac{5}{2} = -1$ $x = 1 - \frac{5}{2} + 2$ $x = \frac{1}{2}$ | $\frac{xy}{z} = \frac{-1}{2} : \frac{5}{2} = \underline{-1/5}$

42-) $\begin{cases} x + 2y + z = 4 \\ ax + 3y - z = 3 \\ 2x - y + z = 2 \end{cases}$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ a & 3 & -1 & 3 \\ 2 & -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -1 & 1 & 2 \\ a & 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -5 & -1 & -6 \\ 0 & 3-2a & -1-a & 3-4a \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -5 & -1 & -6 \\ 0 & 0 & -8-3a & -13-8a \end{bmatrix}$$

$$-8-3a = 0$$
$$-3a = 8$$
$$a = -\frac{8}{3}$$

Se $a = -8/3$ o sistema é incompatível

Se $a \neq -8/3$ o sistema é compatível determinado

43-) $\begin{cases} x + 2y + az = 4 \\ x + 3y - z = 3 \\ 2x - y + z = 2 \end{cases}$

$$\begin{bmatrix} 1 & 3 & -1 & 3 \\ 2 & -1 & 1 & 2 \\ 1 & 2 & a & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -1 & 3 \\ 0 & -7 & 3 & -4 \\ 0 & -1 & a+1 & 1 \end{bmatrix} \xrightarrow[\text{I}]{} \begin{bmatrix} 1 & 3 & -1 & 3 \\ 0 & -7 & 3 & -4 \\ 0 & 0 & 7a+4 & 11 \end{bmatrix}$$

$$7a + 4 = 0$$
$$a = \frac{-4}{7}$$

$\therefore$ Se $a = -4/7$ o sistema é indeterminado

Se $a = 4/7$ o sistema é compatível e determinado

44) $\begin{cases} x + 2y + z = 4 \\ x + 3y - z = 3 \\ 2ax - y + z = 2 \end{cases}$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 3 & -1 & 3 \\ 2a & -1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & -1-4a & 1-2a & 2-8a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1-10a & 1-12a \end{bmatrix}$$

$-1-10a = 0$

$a = -\frac{1}{10}$

Se $a = -1/10$ o Sistema é incompatível

Se $a \neq -1/10$ o Sistema é compatível e determinado

45) $\begin{cases} x + 2y + z = 4 \\ x + 3y - z = 3a \\ 2x - y + z = 2 \end{cases}$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -1 & 1 & 2 \\ 1 & 3 & -1 & 3a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -5 & -1 & -6 \\ 0 & 1 & -2 & 3a-4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -5 & -1 & -6 \\ 0 & 0 & -11 & 15a-26 \end{bmatrix}$$

O sistema é compatível e determinado para $\forall\, a \in \mathbb{R}$

46) $\begin{cases} x + y + az = 1 \\ x + ay + z = a \\ ax + y + z = a^2 \end{cases}$

$$\begin{bmatrix} 1 & 1 & a & 1 \\ 1 & a & 1 & a \\ a & 1 & 1 & a^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & a-1 \\ 0 & 1-a & 1-a^2 & a^2-a \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & a-1 \\ 0 & 1-a & 1-a^2 & a^2-a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & a-1 \\ 0 & 0 & -a^2-a+2 & a^2-1 \end{bmatrix}$$

$-a^2 - a + 2 = 0$ $\qquad$ $a^2 - 1$: $\qquad$ $a = \pm 1$

$a = \dfrac{1 \pm \sqrt{1+8}}{-2}$ $\qquad$ se $a = -2 \Rightarrow (-2)^2 - 1 = 3$

$a = 1 \Rightarrow (1)^2 - 1 = 0$

$\therefore a_1 = \dfrac{1+3}{-2} = -2$

$\therefore$ Se $a = -2$ o sistema é incompativel

$a_2 = \dfrac{1-3}{-2} = 1$ $\qquad$ Se $a = 1$ o sistema é compativel e indeterminado

Se $a \neq 1$ e $a \neq -2$ o sistema é compativel e determinado

47.) $(x_1)CO + (x_2)CO_2 + (x_3)H_2 \rightarrow (x_4)CH_4 + (x_5)H_2O$

$x_1 C + x_2 C = x_4 C \rightarrow x_1 C + x_2 C - x_4 C = 0$

$x_1 O + 2x_2 O = x_5 O \rightarrow x_1 O + 2x_2 O - x_5 O = 0$

$x_3 2H = x_4 . 4H + x_5 . 2H \rightarrow 2x_3 H - 4x_4 H - 2x_5 H = 0$

$$\begin{cases} x_1 + x_2 - x_4 = 0 \\ x_1 + 2x_2 - x_5 = 0 \\ 2x_3 - 4x_4 - 2x_5 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 2 & -4 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 2 & -4 & -2 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_2 + x_4 - x_5 = 0 \\ 2x_3 - 4x_4 - 2x_5 = 0 \end{cases}$$

?

Espaços Vetoriais

63-) R^3: $u=(1,1,1)$; $v=(0,-2,1)$ e $w=(-1,0,1)$

a-) $\frac{t+u}{2} + \frac{v+t}{3} = w \Rightarrow \frac{3t+3u+2v+2t}{6} = w$

$5t+3u+2v=6w \Rightarrow t = \frac{6w-3u-2v}{5}$

a-) $t = \frac{6(-1,0,1)-3(1,1,1)-2(0,-2,1)}{5}$ $\therefore$ $t = \left(\frac{-9}{5}, \frac{1}{5}, \frac{1}{5}\right)$

b-) $z: \begin{cases} x+y+z=u \\ 2x+y+z=w \\ x+y+2z=v \end{cases}$

$$\begin{bmatrix} 1 & 1 & 1 & u \\ 2 & 1 & 1 & w \\ 1 & 1 & 2 & v \end{bmatrix} \longrightarrow \begin{array}{cccc} x & y & z & b \\ \end{array} \begin{bmatrix} 1 & 1 & 1 & u \\ 0 & -1 & -1 & w-2u \\ 0 & 0 & 1 & v-u \end{bmatrix}$$

$\begin{cases} x+y+z=u \\ -y-z=w-2u \\ z=v-u \end{cases}$

$z=(0,-2,1)-(1,1,1)=(-1,-3,0)$

$y=2(1,1,1)-(-1,0,1)-(-1,-3,0)=(4,5,1)$

$x=(1,1,1)-(4,5,1)-(-1,-3,0)=(-2,-1,0)$

$\therefore$ $x=(-2,-1,0)$; $y=(4,5,1)$; $z=(-1,-3,0)$

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64-) $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$

a) $2A - 3B + C = \begin{bmatrix} 5 & 4 \\ 4 & -6 \\ -7 & 1 \end{bmatrix}$

b) $X \in M_{3\times2}$ tal que $\dfrac{X-A}{2} + \dfrac{B-C}{3} = \dfrac{X+A-B}{4}$

$$\frac{6X - 6A + 4B - 4C = 3X + 3A - 3B}{12}$$

$3X = 3A - 3B + 6A - 4B + 4C$

$X = \dfrac{1}{3}(9A - 7B + 4C)$ $\therefore$ $X = \dfrac{1}{3} \cdot \begin{bmatrix} 22 & 12 \\ 11 & -15 \\ -12 & 5 \end{bmatrix}$

c) $\exists\ \alpha, \beta, \in \mathbb{R} \mid C = \alpha A + \beta B$?

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \alpha \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} + \beta \begin{bmatrix} 0 & -1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix}$$

$2\alpha = 1 \Rightarrow \alpha = 1/2$

$\alpha - \beta = -1 \Rightarrow 1/2 + 2 \neq -1$ $\therefore$ Não existe combinação Linear

$-\beta = 2 \Rightarrow \beta = -2$

$-\alpha + 2\beta = 2$

65-) $P_3(\mathbb{R})$ $u = f(t) = 2 - 4t + t^2 - t^3$
$v = g(t) = -1 + t - t^3$
$w = h(t) = 1 + 2t + t^2 + 2t^3$

a) $f(t) = \alpha g(t) + \beta h(t)$?

$$2 - 4t + t^2 - t^3 = \alpha(-1 + t - t^3) + \beta(1 + 2t + t^2 + 2t^3)$$
$$2 - 4t + t^2 - t^3 = -\alpha + \alpha t - \alpha t^3 + \beta + \beta 2t + \beta t^2 + \beta 2t^3$$

$$\begin{cases} -\alpha + \beta = 2 \\ 2\alpha + 2\beta = -4 \quad \mathrm{I} \\ \beta = 1 \quad * \\ -\alpha + 2\beta = -1 \quad \mathrm{II} \end{cases} \qquad \begin{cases} 2\alpha + 2\beta = -4 \\ -\alpha + 2\beta = -1 \end{cases} \sim \begin{cases} 2\alpha + 2\beta = -4 \\ -2\alpha + 4\beta = -2 \end{cases}$$

$$6\beta = -6$$
$$\beta = -1 \quad **$$

$*$ e $**$ são diferentes $\therefore$ eles não são combinação linear

b) $r(t) = f(t) + 2g(t) - H(t)$
$r(t) = 2 - 4t + t^2 - t^3 + 2(-1 + t - t^3) - (1 + 2t + t^2 + 2t^3)$
$r(t) = \cancel{2} - 4t + t^2 - t^3 - \cancel{2} + 2t - 2t^3 - 1 - 2t - t^2 - 2t^3$
$r(t) = -1 - 4t - 9t^3 - 5t^3$

66-) Verifique $W = \{(x, y, z) \mid x + 3y \leq 0\}$ é subespaço vetorial

a) $0 = (0,0,0) \in W$; pois $0 + 30 \leq 0$

b) $\forall w_1, w_2 \in W \Rightarrow (w_1 + w_2) \in W$

$\therefore$ Não é subespaço vetorial

$$\begin{cases} u = (x_1, y_1, z_1) \in \mathbb{R}^3 \mid x_1 + 3y_1 \leq 0 \\ v = (x_2, y_2, z_2) \in \mathbb{R}^3 \mid x_2 + 3y_2 \leq 0 \end{cases}$$

$u + v = (x_1 + x_2, 3(y_1 + y_2), z_1 + z_2) = (\underbrace{x_1 + 3y_1}_{\leq 0}, \underbrace{x_2 + 3y_2}_{\leq 0})$

c) $\lambda w_1 = \lambda(x_1, y_1, z_1) = \lambda x_1, \lambda y_1, z_1$ $\qquad \lambda x + \lambda y = \underbrace{\lambda(x + y)}_{\notin W}$

67-) Verifique se o subconjunto do $\mathbb{R}^3$: $W = \{(x,y,z) / x+y+z \in Q\}$ é subespaço vetorial de $\mathbb{R}^3$

a) $0 \in W$

$0 = (0,0,0) \in W$, pois $0+0+0 \in Q$

b) $\forall w_1, w_2 \in W \Rightarrow (w_1 + w_2) \in W$

$u = (x_1, y_1, z_1) \in W$

$v = (x_2, y_2, z_2) \in W$

$u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_1 + y_2 + z_2, x_2 + y_2 + z_2) \in Q$

c) $\lambda w_1 \in W$

$\lambda (x_1, y_1, z_1) = \lambda x_1, \lambda y_1, \lambda z_1 = \lambda (x_1 + y_1 + z_1) \notin Q$

68-) $V = \{(x,y,z) / x = y\}$ $\quad v = (1,1,3)$

$W = \{(x,y,z) / z = 0\}$ $\quad w = (0,0,0)$

$v + w = (1,1,3) + (0,0,0) = (1,1,3) \notin W$

$\therefore$ Não é subespaço vetorial do $\mathbb{R}^3$

69-) $W = \{(x,y,z) / x+y+z = 0\}$

a) $0 = (0,0,0) \in W$; pois $0+0+0 = 0$

b) $\forall w_1, w_2 \in W \Rightarrow (w_1 + w_2) \in W$

$u = (x_1, y_1, z_1) \in W$ $\quad u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$v = (x_2, y_2, z_2) \in W$ $\quad (\underbrace{x_1 + y_1 + z_1}_{=0}, \underbrace{x_2 + y_2 + z_2}_{=0}) \in W$

c) $\lambda w_1 \in W \Rightarrow (\lambda w_1) \in W$

$\lambda(x_1, y_1, z_1) = (\lambda x_1, \lambda y_1, \lambda z_1) = \lambda(x_1 + y_1 + z_1)$

$x_1 + y_1 + z_1 = 0$

$\therefore$ $W = \{(x,y,z) \mid x+y+z=0\}$ é um subespaço vetorial de $\mathbb{R}^3$

70-) $M_2(\mathbb{R})$: $W = \{A \in M_2(\mathbb{R}) \mid A = A^t\}$ é um subespaço?

a) $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$; pois $0 = 0^t \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

b) $w_1 = \begin{bmatrix} a_1 & c_1 \\ b_1 & d_1 \end{bmatrix}$ $\quad w_2 = \begin{bmatrix} a_2 & c_2 \\ b_2 & d_2 \end{bmatrix}$

$w_1 + w_2 = \begin{bmatrix} a_1 + a_2 & c_1 + c_2 \\ b_1 + b_2 & d_1 + d_2 \end{bmatrix} =$

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75-) Determine x e y para que o vetor $(1, x, 2, y) \in \mathbb{R}^4$
$(1,-1,2,3)$ e $(-1,1,0,2)$

$$(1, x, 2, y) = a(1,-1,2,3) + b(-1,1,0,2)$$

$$\begin{cases} a - b = 1 \\ -a + b = x \\ 2a = 2 \\ 3a + 2b = y \end{cases}$$

$a = 1$
$b = a - 1 = 1 - 1 = 0$

$x = b - a = 0 - 1 = -1$
$y = 3a + 2b = 3 \cdot 1 - 2 \cdot 0 = 3$

$\underline{x = -1 \quad y = 3}$

76-) $p(x) = 2 - x + mx^2 + x^3$
$f(x) = 1 - x - x^2$
$g(x) = -1 - 2x + x^2 - x^3$
$h(x) = 1 + x + x^2 + x^3$

$p(x) = a f(x) + b g(x) + c h(x)$

$$2 - x + mx^2 + x^3 = a(1 - x - x^2) + b(-1 - 2x + x^2 - x^3) + c(1 + x + x^2 + x^3)$$

$$\begin{cases} a - b + c = 2 \\ -a - 2b + c = -1 \\ -a + b + c = m \\ -b + c = 1 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ -1 & -2 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & -3 & 2 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad \begin{cases} a - b + c = 2 \\ -b + c = 1 \\ -c = -2 \end{cases}$$

$c = 2$
$b = 1$
$a = 1$

$-a + b + c = m$
$-1 + 1 + 2 = m = 2$ $\quad \therefore \underline{m = 2}$

77-) $\begin{bmatrix} m & 1 \\ 2 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + c\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$

$m = a + c$

$1 = 2a + b + 3c$

$2 = b + c$

$0 = a + 2b + c$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & 1 & \cdot 1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 7 \end{bmatrix} \quad \begin{cases} 2a + b + 3c = 1 \\ b + c = 2 \\ 4c = 7 \end{cases}$$

$c = 7/4$

$b = \cdot 1/4$

$a = -9/4$

$$m = a + c = \frac{-9}{4} + \frac{7}{4} = \frac{-2}{4} = \frac{-1}{2} \quad \therefore m = \frac{-1}{2}$$

76- $W = \mathbb{R}^3$ $U = \{(x,y,z) \mid x = y\}$ Determine U, V, $U+V$, e $U \cap V$

$V = \{(x,y,z) \mid z = 0\}$

$\forall u = (x,y,z) \in U \Rightarrow v(x,x,z) = x(1,1,0) + z(0,0,1)$,

logo $U = [(1,1,0),(0,0,1)]$

$\forall v = (x,y,z) \in V \Rightarrow v(x,y,0) = x(1,0,0) + y(0,1,0)$, logo

$V = [(1,0,0),(0,1,0)]$

$U + V = [(1,1,0),(0,0,1),(1,0,0),(0,1,0)]$

$U \cap V = \forall i = (x,y,z) \in U \cap V = I = (x,x,0) = x(1,1,0)$ logo

$U \cap V = [(1,1,0)]$

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79-) R^3: $U = [(1,2,1), (-1,0,4)]$ e $W = [(-1,1,0), (1,2,-1)]$

$\forall u = (x,y,z) \in U \Rightarrow (x,y,z) = a(1,2,1) + b(-1,0,4)$; $a, b \in R$

$$\begin{cases} a - b = x \\ 2a = y \\ a + 4b = z \end{cases} \sim \begin{bmatrix} 1 & -1 & x \\ 2 & 0 & y \\ 1 & 4 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & x \\ 0 & 2 & y-2x \\ 0 & 5 & z-x \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & -1 & x \\ 0 & 2 & y-2x \\ 0 & 0 & -8x+5y-2z \end{bmatrix}$$

$5L_2 - 2L_3 \Rightarrow$
$5(y-2x) - 2(z-x)$
$5y - 10x - 2z + 2x =$
$-8x + 5y - 2z$

$\therefore$ A equação homogênea =
$-8x + 5y - 2z = 0$ ou $8x - 5y + 2z = 0$

$\forall w = (x,y,z) \in W \Rightarrow (x,y,z) = a(-1,1,0) + b(1,2,-1)$; $a, b \in R$

$$\begin{cases} -a + b = x \\ a + 2b = y \\ -b = z \end{cases} \sim \begin{bmatrix} -1 & 1 & x \\ 1 & 2 & y \\ 0 & -1 & z \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & x \\ 0 & 3 & x+y \\ 0 & 0 & x+y+3z \end{bmatrix}$$

$\therefore$ A equação homogênea $\Rightarrow$ $x + y + 3z = 0$

Para obter o sistema de geradores de $U \cap W$, resolve o sistema das equações homogêneas

$$\begin{cases} 8x - 5y + 2z = 0 \\ x + y - 3z = 0 \end{cases} \sim \begin{cases} 8x - 5y + 2z = 0 \\ -13y + 22z = 0 \end{cases} \quad x = \frac{-17}{13} \cdot z \quad \therefore y = \frac{-22}{13} \cdot z$$

$$I = \left(\frac{-17}{13}z, \frac{-22}{13}z, z\right) = z\left(\frac{-17}{3}, \frac{-22}{3}, 1\right)$$

$$\therefore\ U \cap W = \left(\frac{-17}{3}, \frac{-22}{3}, 1\right)$$

80-) $\mathbb{R}^4 \Rightarrow (1,-1,2,0), (1,1,3,-1), (0,1,1,2)$

$$(x,y,z,t) = a(1,-1,2,0) + b(1,1,3,-1) + c(0,1,1,2);\ a,b,c \in \mathbb{R}$$

$$\begin{cases} a + b = x \\ -a + b + c = y \\ 2a + 3b + c = z \\ -b + 2c = t \end{cases} \sim \begin{bmatrix} 1 & 1 & 0 & x \\ -1 & 1 & 1 & y \\ 2 & 3 & 1 & z \\ 0 & -1 & 2 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 2 & 1 & x+y \\ 0 & 1 & 1 & z-2x \\ 0 & -1 & 2 & t \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 2 & 1 & x+y \\ 0 & 0 & -1 & 5x+y-2z \\ 0 & 0 & 5 & x+y+2t \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 2 & 1 & x+y \\ 0 & 0 & -1 & 5x+y-2z \\ 0 & 0 & 0 & 26x+6y-10z+2t \end{bmatrix}$$

$$26x + 6y - 10z + 2t = 0$$

$$W: 13x + 3y - 5z + t = 0$$

81) $\mathbb{R}^3$, $U = [(1,2,1), (2,1,0)]$

$$\forall u = (x,y,z) \in U \Rightarrow (x,y,z) = a(1,2,1) + b(2,1,0);\ a,b \in \mathbb{R}$$

$$\begin{cases} a + 2b = x \\ 2a + b = y \\ a = z \end{cases} \longrightarrow \begin{bmatrix} 1 & 2 & x \\ 2 & 1 & y \\ 1 & 0 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & x \\ 0 & -3 & y-2x \\ 0 & -2 & z-x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & x \\ 0 & -3 & y-2x \\ 0 & -2 & z-x \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & x \\ 0 & -3 & y-2x \\ 0 & 0 & -x+2y-3z \end{bmatrix}$$

Eq. homogenea que define U: $-x+2y-3z=0$

82-) $\mathbb{R}^4$: $U=[(1,1,0,-1);(1,2,3,0);(2,3,3,-1)]$

$V=[(1,2,2,-2);(2,3,2,-3);(1,3,4,-3)]$

Sistema gerador de $U \cap V = ?$

$\forall u=(x,y,z,t) \in U \Rightarrow u=(x,y,z,t)=a(1,1,0,-1)+b(1,2,3,0)+c(2,3,3,-1)$

$$\begin{cases} a+b+2c=x \\ a+2b+3c=y \\ 3b+3c=z \\ -a-c=t \end{cases} \sim \begin{bmatrix} 1 & 1 & 2 & x \\ 1 & 2 & 3 & y \\ 0 & 3 & 3 & z \\ -1 & 0 & -1 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 1 & 1 & y-x \\ 0 & 3 & 3 & z \\ 0 & 1 & 1 & x+t \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 1 & 1 & y-x \\ 0 & 0 & 0 & 3x-3y+z \\ 0 & 0 & 0 & x+t+x-y \end{bmatrix} \qquad \begin{cases} 2x-y+t=0 \\ 3x-3y+z=0 \end{cases}$$

$\forall v=(x,y,z,t) \in V \Rightarrow v=(x,y,z,t)=a(1,2,2,-2)+b(2,3,2,-3)+c(1,3,4,-3)$

$$\begin{cases} a+2b+c=x \\ 2a+3b+3c=y \\ 2a+2b+4c=z \\ -2a-3b-3c=t \end{cases} \sim \begin{bmatrix} 1 & 2 & 1 & x \\ 2 & 3 & 3 & y \\ 2 & 2 & 4 & z \\ -2 & -3 & -3 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & x \\ 0 & -1 & 1 & y-2x \\ 0 & -2 & 2 & z-2x \\ 0 & 1 & -1 & t+2x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & x \\ 0 & -1 & 1 & y-2x \\ 0 & 0 & 0 & 2x-2y+z \\ 0 & 0 & 0 & y+t \end{bmatrix} \qquad \begin{cases} 2x-2y+z=0 \\ y+t=0 \end{cases}$$

Resolvendo o seguinte sistema:

$$\begin{cases} 2x-y+0+t=0 \\ 3x-3y+z+0=0 \\ 2x-2y+z+0=0 \end{cases} ; \ t=-y \quad \sim \begin{cases} 2x-2y=0 \\ 3x-3y+z=0 \\ 2x-2y+z=0 \end{cases}$$

$x=y \qquad 3x-3y+z=0$

$\qquad 3y-3y+z=0 \ \therefore z=0$

$(x,y,z,t)=(y,y,0,-y)=y(1,1,0,-1)$

$\therefore \underline{U \cap V = (1,1,0,-1)}$

83-) $w=(x,y,z,t) \in W \Rightarrow$

$\Rightarrow (x,y,z,t)=a(1,2,0,3)+b(1,1,1,4)+c(1,0,2,5) \ ; a,b,c \in \mathbb{R}$

$$\begin{cases} a+b+c=x \\ 2a+b=y \\ b+2c=z \\ 3a+4b+5c=t \end{cases} \sim \begin{bmatrix} 1 & 1 & 1 & x \\ 2 & 1 & 0 & y \\ 0 & 1 & 2 & z \\ 3 & 4 & 5 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & x \\ 0 & -1 & -2 & y-2x \\ 0 & 1 & 2 & z \\ 0 & 1 & 2 & t-3x \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & x \\ 0 & -1 & -2 & y-2x \\ 0 & 0 & 0 & -2x+y+z \\ 0 & 0 & 0 & -5x+y+t \end{bmatrix}$$

$-2x+y+z=0$ e $-5x+y+t=0$

84-) $S = \{(x,y,z) \mid x - 2y - t = 0 \,;\, x + z + t = 0\}$

$$\begin{cases} x - 2y - t = 0 \\ x + z + t = 0 \end{cases} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 2 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{cases} x - 2y - t = 0 \\ 2y + z + 2t = 0 \end{cases} \qquad \begin{matrix} x = t + 2y \\ z = -2t - 2y \end{matrix}$$

$$(x,y,z) = (t + 2y \,;\, y \,;\, -2t - 2y \,;\, t)$$

$$= t(1,0,-2,1) + y(2,1,-2,0)$$

$$S = [(1,0,-2,1) \,;\, (2,1,-2,0)]$$

85.) $(-1,0,3)$ e $(2,1,-1)$ $\quad v=(1,2,k)$

$u=(x,y,z)\in U \Rightarrow u=(x,y,z)=a(-1,0,3)+b(2,1,-1);\ a,b\in\mathbb{R}$

$$\begin{cases}-a+2b=x\\ b=y\\ 3a-b=z\end{cases} \sim \begin{bmatrix}-1 & 2 & x\\ 0 & 1 & y\\ 3 & -1 & z\end{bmatrix} \rightarrow \begin{bmatrix}-1 & 2 & x\\ 0 & 1 & y\\ 0 & 5 & z+3x\end{bmatrix} \rightarrow \begin{bmatrix}-1 & 2 & x\\ 0 & 1 & y\\ 0 & 0 & 3x-5y+z\end{bmatrix}$$

$3x-5y+z=0$ $\quad \rightarrow \quad 3.1-5.2+k=0$

$v=(1,2,k)$ $\qquad \underline{k=7}$

86.) $U\cap T=?$ $\quad U=[(1,0,0)+(1,1,1)]$ $\quad T=[(1,-1,2),(0,1,-1)]$

$\forall u=(x,y,z)\in U \therefore u=(x,y,z)=a(1,0,0)+b(1,1,1);\ a,b\in\mathbb{R}$

$$\begin{cases}a+b=x\\ b=y\\ b=z\end{cases} \sim \begin{bmatrix}1 & 1 & x\\ 0 & 1 & y\\ 0 & 1 & z\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & x\\ 0 & 1 & y\\ 0 & 0 & z-y\end{bmatrix} \therefore z-y=0$$

$\forall t=(x,y,z)\in T \therefore t=(x,y,z)=a(1,-1,2)+b(0,1,-1);\ a,b\in\mathbb{R}$

$$\begin{cases}a=x\\ -a+b=y\\ 2a-b=z\end{cases} \sim \begin{bmatrix}1 & 0 & x\\ -1 & 1 & y\\ 2 & -1 & z\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & x\\ 0 & 1 & x+y\\ 0 & -1 & z-2y\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & x\\ 0 & 1 & x+y\\ 0 & 0 & x-y+z\end{bmatrix}$$

$\therefore x-y+z=0$

$$\begin{cases}x-y+z=0\\ z-y=0\end{cases} \quad \therefore z=y \qquad \begin{aligned}x&=y+z\\ x&=y+y=2y\end{aligned}$$

$I=(2y,y,y)=y(2,1,1)$ $\quad \therefore \underline{U\cap T=(2,1,1)}$

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87-) $(1,-1,3)$; $(0,1,-1)$; $(2,3,1)$

$(x,y,z) = a(1,-1,3) + b(0,1,-1) + c(2,3,1)$; $a,b,c \in \mathbb{R}$

$$\begin{cases} a + 0 + 2c = x \\ -a + b + 3c = y \\ 3a - b + c = z \end{cases} \sim \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

O conjunto de vetores é LD, portanto __não__ gera $\mathbb{R}^3$

88-) $U = \{(x,y,z) \mid x + 2y + 3z = 0\}$ Determine um sist. gerador
$W = \{(x,y,z) \mid 3x - y - z = 0\}$ de $U, W, U \cap W$ e $U + W$

$\forall u = (x,y,z) \in U \Rightarrow u = (x,y,z) = (-2y - 3z, y, z) =$
$= y(-2, 1, 0) + z(-3, 0, 1)$
$\therefore U = [(-2,1,0) ; (-3,0,1)]$

$\forall w = (x,y,z) \in W \Rightarrow w = (x,y,z) = (x ; 3x - z, z)$
$= x(1, 3, 0) + z(0, -1, 1)$
$W = [(1,3,0) ; (0,-1,1)]$

$\forall i = (x,y,z) \in U \cap W$

$$\begin{cases} x + 2y + 3z = 0 \\ 3x - y - z = 0 \end{cases} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & -1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -7 & -10 & 0 \end{bmatrix}$$

$$\begin{cases} x + 2y + 3z = 0 \\ -7y - 10z = 0 \end{cases} \quad y = \frac{10z}{-7} \quad x = -3z - 2y = \frac{-1}{7}z$$

89.) $S=\{(x,y,z,t) \mid 2x-y=z+t=0\}$ $\quad T, S, S\cap T$

$T=\{(x,y,z,t) \mid x-2y-z+2t=0\}$

$\forall s=(x,y,z)\in S \quad s=(x,y,z,t)=(x,2x,-t,t)$

$= x(1,2,0,0)+t(0,0,-1,1)$

$\begin{cases} 2x-y=0 & \therefore y=2x \\ z+t=0 & \therefore z=-t \end{cases}$ $\quad S=[(1,2,0,0);(0,0,-1,1)]$

$\forall t=(x,y,z)\in T \quad t=(x,y,z,t)=(2y+z-2t,y,z,t)$

$=y(2,1,0,0)+z(1,0,1,0)+t(-2,0,0,1)$

$x-2y-z+2t=0$

$x=2y+z-2t \qquad T=[(2,1,0,0);(1,0,1,0);(-2,0,0,1)]$

$\forall i=(x,y,t)\in I=S\cap T$

$\begin{cases} 2x-y=0 \\ z+t=0 \\ x-2y-z+2t=0 \end{cases} \sim \begin{bmatrix} 1 & -2 & -1 & 2 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -1 & 2 \\ 0 & 3 & 2 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\begin{cases} x-2y-z+2t=0 \\ 3y+2z-4t=0 \\ z+t=0 \end{cases}$ $\quad \begin{aligned} z&=-t \\ 3y-2t-4t&=0 \\ y&=2t \end{aligned}$ $\quad \begin{aligned} x-4t+t+2t&=0 \\ x&=t \end{aligned}$

$\therefore (x,y,z,t)=(t,2t,-t,t)=t(1,2,-1,1)$

$S\cap T=[(1,2,-1,1)]$

90-) $u=(2,1,0)$ $v=(1,-1,2)$ $w=(0,3,-4)$

$(x,y,z)=m(2,1,0)+n(1,-1,2)+t(0,3,-4)$

$$\begin{cases} 2m+n=x \\ m-n+3t=y \\ 2n-4t=z \end{cases} \sim \begin{bmatrix} 2 & 1 & 0 & x \\ 1 & -1 & 3 & y \\ 0 & 2 & -4 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 0 & x \\ 0 & 3 & -6 & x-2y \\ 0 & 2 & -4 & z \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & x \\ 0 & 3 & -6 & x-2y \\ 0 & 0 & 0 & -2x+4y+3z \end{bmatrix}$$

$\therefore -2x+4y+3z=0$

$-2a+4b+3c=0$ ou $2a-4b-3c=0$

41-) $u_1=(1,0,-1,3)$ $u_2=(-1,1,2,0)$ $u_3=(0,2,1,3)$

$(x,y,z,t)=a(1,0,-1,3)+b(-1,1,2,0)+c(0,2,1,3)$; $a,b,c\in\mathbb{R}$

$$\begin{cases} a-b=x \\ b+2c=y \\ -a+2b+c=z \\ 3a+3c=t \end{cases} \sim \begin{bmatrix} 1 & -1 & 0 & x \\ 0 & 1 & 2 & y \\ -1 & 2 & 1 & z \\ 3 & 0 & 3 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & x \\ 0 & 1 & 2 & y \\ 0 & 1 & 1 & x+z \\ 0 & 3 & 3 & -3x+t \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & x \\ 0 & 1 & 2 & y \\ 0 & 0 & -1 & x+z-y \\ 0 & 0 & -3 & -3x+t-3y \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & x \\ 0 & 1 & 2 & y \\ 0 & 0 & -1 & x+z-y \\ 0 & 0 & 0 & -6x-3z+t \end{bmatrix}$$

Equação homogênea: $-6x - 3z + t = 0$ ou $6x + 3z - t = 0$

$w = (0,-1,1,3)$ $\in$ subespaço gerado = ?

$6x + 3z - t = 0$ $\therefore$ O vetor $\vec{w}$ $\in$ ao subespaço gerado

$6.0 + 3.1 - 3 = 0$

92-) $P_1(x) = 1$ $P_2(x) = 1-x$ $P_3(x) = 1-x+x^2$ $P_4(x) = 1+x+x^3$

$$a(1-x) + b(1-x+x^2) + c(1+x+x^3) = (0 + 0x + 0x^2 + 0x^3)$$

$$\begin{cases} a + b + c = 0 \\ -ax - bx + cx = 0 \\ bx^2 = 0 \\ cx^3 = 0 \end{cases} \sim \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} ?$$

Fazer depois

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93.)

$a + b(1-t) + c(1-2t+t^2) + d(1-3t+3t^2-t^3) = 0 + 0t + 0t^2 + 0t^3$

$$\begin{cases} a = 0 \\ b - bt = 0 \\ c - 2ct + ct^2 = 0 \\ d - 3td + 3t^2d - dt^3 = 0 \end{cases} \sim \begin{pmatrix} 1 & -3 & 3 & -1 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -3 & 1 \\ 0 & 3 & -3 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Os polinômios é linearmente Independente, com isso ele é um espaço vetorial.

94.) $S_1 = [(1,2,0);(3,1,2)]$ $S_2 = [(0,1,1);(1,3,0)]$

$\forall v_1 = (x,y,z) \in S_1$:

$(x,y,z) = a(1,2,0) + b(3,1,2)$; $a,b \in \mathbb{R}$

$$\begin{cases} a + 3b = x \\ 2a + b = y \\ 2b = z \end{cases} \sim \begin{bmatrix} 1 & 3 & x \\ 2 & 1 & y \\ 0 & 2 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & x \\ 0 & -5 & y-2x \\ 0 & 0 & -4x+2y+5z \end{bmatrix}$$

A equação homogênea de S_1 é: $-4x + 2y + 5z = 0$

$\forall v_2 = (x,y,z) \in S_2$

$\therefore (x,y,z) = a(0,1,1) + b(1,3,0)$; $a,b \in \mathbb{R}$

$$\begin{cases} b = x \\ a + 3b = y \\ a = z \end{cases} \sim \begin{bmatrix} 1 & 3 & y \\ 0 & 1 & x \\ 1 & 0 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & y \\ 0 & 1 & x \\ 0 & -3 & z-y \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & y \\ 0 & 1 & x \\ 0 & 0 & 3x-y+z \end{bmatrix}$$

$S_1 \cap S_2$: $\begin{cases} -4x+2y+5z=0 \\ 3x-y+z=0 \end{cases} \sim \begin{cases} -4x+2y+5z=0 \\ 6x-2y+2z=0 \end{cases}$

$\begin{cases} -4x+2y+5z=0 \\ 2x+7z=0 \end{cases}$ $\quad x = \frac{-7}{2}z \quad y = \frac{4x-5z}{2} = \frac{-14z-5z}{2} = \frac{-19z}{2}$

$(x,y,z) = \left(\frac{-7}{2}z, \frac{-19}{2}z, z\right) = (-7z, -19z, 2z) = z(-7,-19,2)$

$S_1 \cap S_2 = [(-7,-19,2)]$

95-) $a\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} + d\begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{cases} a+b+0+d=0 \\ a+\ +c+d=0 \\ 0\ 2b+3c+3d=0 \\ a+b+c\ 0=0 \end{cases} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 3 & 3 \\ 1 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow$

$\longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 8 \end{pmatrix}$ Não é combinação Linear $\therefore$ Gera espaço vetorial.

92-) $S = \{(-1,1,-3,4,1), (1,0,-1,2,-3), (1,-2,5,2,7), (2,-1,1,m,-1)\}$

$\begin{pmatrix} -1 & 1 & -3 & 4 & 1 \\ 1 & 0 & -1 & 2 & -3 \\ 1 & -2 & 5 & 2 & 7 \\ 2 & -1 & 1 & m & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & -3 & 4 & 1 \\ 0 & 1 & -4 & 6 & -2 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 1 & -5 & m+8 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & -3 & 4 & 1 \\ 0 & 1 & -4 & 6 & -2 \\ 0 & 0 & -2 & 12 & 6 \\ 0 & 0 & -1 & m-2 & 3 \end{pmatrix}$

$$\begin{bmatrix} -1 & 1 & -3 & 4 & 1 \\ 0 & 1 & -4 & 6 & -2 \\ 0 & 0 & -2 & 12 & 6 \\ 0 & 0 & -1 & m-2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & -3 & 4 & 1 \\ 0 & 1 & -4 & 6 & -2 \\ 0 & 0 & -2 & 12 & 6 \\ 0 & 0 & 0 & 8-m & 0 \end{bmatrix}$$

Para que o conjunto seja linearmente dependente, a sua ultima linha tem que ser 0,0,0,0,0 $\therefore 8-m=0 \therefore m=8$

93-) $S = \{x^3+3x^2-2x+1,\ x^2+x,\ -x^3+x+1,\ x^3+8x^2-3x+4\}$

$a(x^3+3x^2-2x+1)+b(x^2+x)+c(-x^3+x+1)+d(x^3+8x^2-3x+4)=0$

$$\begin{cases} a+0+c+4d=0 \\ -2a+b+c-3d=0 \\ 3a+b+0+8d=0 \\ a+0-c+d=0 \end{cases} \sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ -2 & 1 & 1 & -3 \\ 3 & 1 & 0 & 8 \\ 1 & 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -11 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & -2 & -3 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -11 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

O conjunto é linearmente dependente

94-) $a(u+v)+b(v+w)+c(u+w)=0$

$au+av+bv+bw+cu+cw$

$u(a+c)+v(a+b)+w(b+c)=0$

$\star$ Como (u,v,w) é L.I logo $\{u+v, v+w, u+w\}$ é L.I

95-) $\{1+x+x^2, -2x+x^2, 1+2x-2x^2\}$

$a(1+x+x^2)+b(-2x+x^2)+c(1+2x-2x^2)=0$

$a+ax+ax^2-2bx+bx^2+c+2cx-2cx^2=0$

$(a+c)+x(a-2b+2c)+x^2(a+b-2c)=0+0x+0x^2$

$$\begin{cases} a + c = 0 \\ a-2b+2c = 0 \\ a+b-2c=0 \end{cases} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{cases} a+b-2c=0 \\ -3b+4c=0 \\ -5c=0 \end{cases} \therefore a=b=c=0$$

$\therefore$ O conjunto de vetores é LI

96-) $\{u+w, u+v, u-v-w\}$

$a(u+w)+b(u+v+c(u-v-w)=0$

$au+aw+bu+bv+cu-cv-cw=0$

$u(a+b+c)+v(b-c)+w(a-c)=0$

$$\begin{cases} a+b+c=0 \\ b-c=0 \\ a -c=0 \end{cases} \sim \begin{cases} a+b+c=0 \\ b-c=0 \\ -b+c=0 \end{cases} \begin{cases} a+b+c=0 \\ 0+0=0 \\ 0+0=0 \end{cases}$$

$\therefore$ O conjunto de vetores é linearmente independente

97-) $\{-2, -1+x, x+x^2\}$

$a(-2) + b(-1+x) + c(x+x^2) = 0$

$-2a - b + bx + cx + cx^2 = 0$

$(-2a - b) + (b+c)x + cx^2 = 0$

$\begin{cases} -2a - b = 0 \\ b + c = 0 \\ c = 0 \end{cases}$ $\therefore$ O conjunto é linearmente independente

98-) $\{f, g, h\} \Rightarrow \{1, e^t, e^{2t}\}$

$a + be^t + ce^{2t} = 0$

Admitindo $t=0$, $t=1$, $t=2$

$\begin{cases} a + b + c = 0 \\ a + be + ce^2 = 0 \\ a + be^2 + ce^4 \end{cases} \sim \begin{cases} a + b + c = 0 \\ a + b + c = 0 \\ a + b + c = 0 \end{cases}$ $\therefore$ O conjunto de vetores é LI

99-) $V = C[0, \pi]$ $\{1, \operatorname{sen} x, \cos x\}$

$a + b \operatorname{sen} x + c \cos x = 0$

?

100-) $\{(m, n, 3); (2, m-n, 1)\}$

$a(m, n, 3) + b(2, m-n, 1) = 0$

$\begin{cases} am + 2b = 0 \\ an + bm - bn = 0 \\ 3a + b = 0 \end{cases} \sim \begin{bmatrix} m & 2 & 0 \\ n & m-n & 0 \\ 3 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} m & 2 & 0 \\ 0 & m^2-mn-2n & 0 \\ 0 & m-6 & 0 \end{bmatrix}$

$$\begin{bmatrix} m & 2 & 0 \\ 0 & m^2-mn-2n & 0 \\ 0 & m-6 & 0 \end{bmatrix}$$

$am + 2b = 0$

$b(m-6) = 0$

$bm - 6b = 0$

$m = \frac{6\cancel{b}}{\cancel{b}} = 6$

$m^2 - mn - 2n = 0$

$m = 6$

$6^2 - 6n - 2n = 0 \Rightarrow 36 - 6n - 2n = 0$

$36 - 8n = 0$

$n = \frac{36}{8} = \frac{9}{2}$

$m = 6 \quad n = \frac{9}{2}$

101-) $\{ u+v-w,\ u-v+w,\ -u+v+w \}$

$a(u+v-w) + b(u-v+w) + c(-u+v+w) = 0 \ ; \ a, b, c \in \mathbb{K}$

$au + av - aw + bu - bv + bw - cu + cv + cw = 0$

$u(a+b-c) + v(a-b+c) + w(-a+b+c) = 0$

$$\begin{cases} a + b - c = 0 \\ a - b + c = 0 \\ -a + b + c = 0 \end{cases}$$

$+ \Rightarrow a = 0 \qquad \text{I} \quad a + b - c = 0$

$+ \Rightarrow c = 0 \qquad 0 + b - 0 = 0 \quad b = 0$

$\therefore$ O conjunto de vetores é L.I

102-) $\{ (3, a^2-4, 6) ; (1, a, b) \}$

$x(3, a^2-4, 6) + y(1, a, b) = 0, \ x, y \in \mathbb{R}$

$$\begin{cases} 3x + y = 0 \\ a^2x - 4x + ay = 0 \\ 6x + by = 0 \end{cases} \sim \begin{bmatrix} 3 & 1 & 0 \\ a^2-4 & a & 0 \\ 6 & b & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & a^2-3a-4 & 0 \\ 0 & b-2 & 0 \end{bmatrix}$$

$a^2 - 3a - 4 = 0 \Rightarrow \Delta = 25 \ \begin{cases} a_1 = 4 \\ a_2 = -1 \end{cases} \qquad b = 2$

R: $a = 4$ e $b = 2$ ou $a = -1$ e $b = 2$

/ /

103-) $\{(m,1,2);(1,1,3);(0,-1,1)\}$

$a(m,1,2)+b(1,1,3)+c(0,-1,1)=0$; $a,b,c \in R$

$$\begin{cases} am+b=0 \\ a+b-c=0 \\ 2a+3b+c=0 \end{cases} \qquad \begin{cases} a+b-c=0 \\ 2a+3b+c=0 \end{cases} \qquad am+b=0$$

$3a+4b=0$ $\qquad -b=am$

$b=\frac{-3a}{4}$ $\qquad b=-am$

$-\frac{3}{4}a=-am$

$\therefore m \neq \frac{3}{4}$ para ser L.I

104-) $\{(1,2,1,0);(2,m,0,0);(1,3,3,0)\}$

$a(1,2,1,0)+b(2,m,0,0)+c(1,3,3,0)=0$; $a,b,c \in R$

$$\begin{cases} a+2b+c=0 \\ 2a+am+3c=0 \\ a+3c=0 \end{cases} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & m & 3 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & m-4 & 1 & 0 \\ 0 & -2 & 2 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & m-4 & 1 & 0 \\ 0 & 0 & 2m-6 & 0 \end{bmatrix}$$

$2m-6=0$

$m=\frac{6}{2}=3$

Para que o conjunto de vetores possa ser linearmente dependente $m=3$

(105-) $\{(1,-1,2);(3,0,1);(0,-1,-1);(-1,3,4)\}$

$a(1,-1,2)+b(3,0,1)+c(0,-1,-1)+d(-1,3,4)=0 \; ; a,b,c,d \in \mathbb{R}$

$$\begin{cases} a+3b+0-d=0 \\ -a+0-c+3d=0 \\ 2a+b-c+4d=0 \end{cases} \sim \begin{bmatrix} 1 & 3 & 0 & -1 \\ -1 & 0 & -1 & 3 \\ 2 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 3 & -1 & 2 \\ 0 & -5 & -1 & 6 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -8 & 30 \end{bmatrix} \sim \begin{cases} a+3b-d=0 \\ 3b-c+2d=0 \\ -8c+30d=0 \end{cases}$$

?

(106-) $\{(1,1,-1,3);(0,1,-2,1);(0,2,1,5)\}$

$a(1,1,-1,3)+b(0,1,-2,1)+c(0,2,1,5)=(0,0,0,0) \; ; a,b,c \in \mathbb{R}$

$$\begin{cases} a=0 \\ a+b+2c=0 \\ -a-2b+c=0 \\ 3a+b+5c=0 \end{cases} \quad \begin{bmatrix} 3 & 1 & 5 \\ -1 & -2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 5 \\ 0 & -5 & 8 \\ 0 & -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 5 \\ 0 & -5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$$

Conjunto L.I.

(107-) $\{1-x,\; 1-x-x^2,\; 2x+x^2\}$

$a(1-x)+b(1-x-x^2)+c(2x+x^2)=0+0x+0x^2$

$$\begin{cases} a+b+0=0 \\ -a-b+2c=0 \\ 0-b+c=0 \end{cases} \quad \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

?

(108-) $\{(1,-1,2,1);(0,1,-3,2);(-1,5,0,1);(2,1,m,-1)\}$

$$a(1,-1,2,1)+b(0,1,-3,2)+c(-1,5,0,1)+d(2,1,m,-1)=(0,0,0,0)$$

$$\begin{cases} a+0-c+2d=0 \\ -a+b+5c+d=0 \\ 2a-3b+0+dm=0 \\ a+2b+c-d=0 \end{cases} \sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 5 & 1 \\ 2 & -3 & 0 & m \\ 1 & 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 5 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 3 & 6 & 0 \\ 0 & -1 & 10 & m+2 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} -1 & 1 & 5 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & -6 & -9 \\ 0 & 0 & 14 & m+5 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 5 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & -6 & -9 \\ 0 & 0 & 0 & 6m-96 \end{pmatrix}$$

$6m-96=0$

$m=\frac{96}{6}=16$

$\therefore m \neq 16$

(109) $\{(-1,1,-3,\alpha,1);(2,-1,1,2,-1),(1,0,-1,2,-3);(1,-2,5,2,7)\}$

$$a(2,-1,1,2,-1)+b(1,0,-1,2,-3)+c(1,-2,5,2,7)+d(-1,1,-3,\alpha,1)=(0,0,0,0,0)$$

$$\begin{cases} 2a+b+c-d=0 \\ -a+0-2c+d=0 \\ a-b+5c-3d=0 \\ 2a+2b+2c+\alpha=0 \\ -a-3b+7c+d=0 \end{cases} \begin{pmatrix} 2 & 1 & 1 & -1 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 5 & -3 \\ -1 & -3 & 7 & 1 \\ 2 & 2 & 2 & \alpha \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 1 & -1 \\ 0 & 1 & -3 & 1 \\ 0 & 3 & -9 & 5 \\ 0 & -5 & & \\ & & & \end{pmatrix}$$

Base de um espaço vetorial finitamente gerado

Ex 1: $B = \{v_1 = (1,0) ; v_2 = (0,1)\}$ é uma base do $\mathbb{R}^2$?

i) $\forall (x,y) \in \mathbb{R}^2$, se $(x,y) = \alpha(1,0) + \beta(0,1)$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \therefore LI$$

Ex 2: $B = \{v_1 = (1,0,0) ; v_2 = (0,1,0), v_3 = (0,0,1)\}$ é uma base do $\mathbb{R}^3$

i) $\forall b = (x,y,z) \in \mathbb{R}^3$

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1) ; x,y,z \in \mathbb{R}$

ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \therefore$ Conjunto L.I.

111.) $\{(1,t,1);(0,1,t),(t,1,0)\}$ seja uma base do $\mathbb{R}^3$

i $\begin{bmatrix} 1 & t & 1 \\ t & 1 & 0 \\ 0 & 1 & t \end{bmatrix} \rightarrow \begin{bmatrix} 1 & t & 1 \\ 0 & 1-t^2 & -t \\ 0 & 0 & t^3-2t^2 \end{bmatrix}$ $\quad t^3 - t^2 = 0$, $\quad t(t^2 - 2t^2) = 0$ $\quad t \neq 0$, $t \neq \pm\sqrt{2}$

112.) $\{(1,0,1); (-1,a,a); (0,a^2,1)\}$ é base do $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & a & a \\ 0 & a^2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & a+1 \\ 0 & a^2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & a+1 \\ 0 & 0 & -a^2-a+1 \end{bmatrix} \quad a \neq \frac{1 \pm \sqrt{5}}{2}$$

(113-) $S = [(1,-1,2);(0,1,-1)]$ $\quad T = [(-1,3,1);(2,1,0)]$

$\forall s = (x,y,z) \in S$

$\therefore (x,y,z) = a(1,-1,2) + b(0,1,-1) \; ; a,b \in \mathbb{R}^3$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \therefore LI$$

$$\begin{cases} a = x \\ -a + b = y \\ 2a - b = z \end{cases} \quad \begin{bmatrix} 2 & -1 & z \\ -1 & 1 & y \\ 1 & 0 & x \end{bmatrix} \to \begin{bmatrix} 2 & -1 & z \\ 0 & 1 & 2y+z \\ 0 & -1 & z-2x \end{bmatrix} \to \begin{bmatrix} 2 & -1 & z \\ 0 & 1 & 2y+z \\ 0 & 0 & -2x+2y+2z \end{bmatrix}$$

Eq. homogénea $= -2x + 2y + 2z = 0$

$\forall t = (x,y,z) \in S$

$\therefore (x,y,z) = a(-1,3,1) + b(2,1,0) \; ; a,b \in \mathbb{R}^3$

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 3 & 1 \\ 0 & 7 & 2 \end{bmatrix} \quad \therefore \text{é } LI$$

$$\begin{cases} -a + 2b = x \\ 3a + b = y \\ a = z \end{cases} \sim \begin{bmatrix} -1 & 2 & x \\ 3 & 1 & y \\ 1 & 0 & z \end{bmatrix} \to \begin{bmatrix} -1 & 2 & x \\ 0 & 7 & y+3x \\ 0 & 2 & z+y \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} -1 & 2 & x \\ 0 & 7 & y+3x \\ 0 & 2 & -6x+5y+7z \end{bmatrix}$$

Eq. homogénea : $-6x + 5y + 7z = 0$

$$\begin{cases} -6x + 5y + 7z = 0 \\ -x + y + z = 0 \end{cases} \sim \begin{cases} -6x + 5y + 7z = 0 \\ 0 - y + z = 0 \end{cases} \quad y = z$$

$$x = \frac{-5z + 7z}{-6}$$

$x = \frac{2z}{-6} = -\frac{1}{3}z$ $\quad (x,y,z) = (-\frac{1}{3}z, z, z) = z(-\frac{1}{3}, 1, 1)$

$S \cap T = (-\frac{1}{3}, 1, 1)$,

114-) $U = \{(a,b,c,d) \mid a+c-d=0\}$

a) $W = \{(a,b,c,d) \mid a+d=0 ; c-2b=0\}$

$\forall w = (x,y,z,t) \in W$ $\quad a+d=0 \quad a=-d$

$\quad c-2b=0 \quad c=2b$

$(x,y,z,t) = (-d, b, 2b, d)$

$= d(-1,0,0,1) + b(0,1,2,0) \; ; \; d,b \in \mathbb{R}$

$\therefore W = \{(-1,0,0,1) ; (0,1,2,0)\}$

dim = 2

b) $U \cap W$:

$$\begin{cases} a+c-d=0 \\ a+d=0 \\ -2b+c=0 \end{cases} \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & -2 & 1 & 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \sim \begin{cases} a+c-d=0 \\ -2b+c=0 \\ -c+2d=0 \end{cases}$$

$\therefore c = 2d \quad \therefore 2d = 2b$

$c = 2b \quad d = b$

$a + 2b - b = 0$

$a = -b$

$(x,y,z,t) = (-b, b, 2b, b) = b(-1,1,2,1)$

$U \cap W = (-1,1,2,1) \quad \dim(U \cap W) = 1$

(115-) $S_1 = [(1,2,3,0);(2,3,3,-1);(0,1,3,-1)]$
$S_2 = [(1,2,2,-2);(1,3,4,-3);(0,-1,-2,1)]$

$S_1 + S_2$ $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 0 & 1 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 1 & 3 & 4 & -3 \\ 0 & -1 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -3 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -2 & 1 \end{pmatrix} \to$

$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$B = (g_1 = (1,2,3,0);\ g_2 = (0,-1,-3,-1);\ g_3 = (0,0,-1,-2);\ g_4 = (0,0,0,-2))$

$\dim(S_1 + S_2) = 4$

116-) $S = \{(x,y,z,t) \mid x - z + t = 0;\ x - 3z = 0\}$
$T = [(1,0,5,-1);(2,1,0,3)]$

$\forall s = (x,y,z,t) \in S$ $\quad x - z + t = 0$
$x - 3z = 0$

$x = 3z$
$t = z - x = z - 3z = -2z$ $\quad \therefore (x,y,z,t) = (3z, y, z, -2z)$

$= z(3,0,1,-2) + y(0,1,0,0)$

$\therefore S = [(3,0,1,-2) ; (0,1,0,0)]$

$\forall t = (x,y,z,t) \in T$

$T = [(1,0,5,-1) ; (2,1,0,3)]$

a)

$\therefore \quad S+T = [(3,0,1,-2) ; (0,1,0,0) ; (1,0,5,-1) ; (2,1,0,3)]$

$\dim(S+T) = 4$

b) $\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$

$\dim(S \cap T) = \dim S + \dim T - \dim(S+T)$

$\dim(S \cap T) = 2 + 2 - 4 = 0$

$\therefore$ A base de $(S \cap T) = \emptyset$

117.) $T = \{(x,y,z) \mid x - y + \lambda z = 0 \text{ e } y + z = 0\}$

$S = \{(x,y,z) \mid x + y + z = 0\}$

$\forall t = (a,b,c) \in T \qquad \begin{cases} x - y + \lambda z \\ y + z = 0 \end{cases} \qquad y = -z$

$x = \lambda z + y = \lambda z - z = z(\lambda - 1)$

$\therefore (a,b,c) = (z(\lambda-1), -z, z) = z(\lambda-1, -1, 1)$

$T = \{(\lambda-1, -1, 1)\} \qquad \dim T = 1$

$\forall s = (a,b,c) \in S \qquad x + y + z = 0 \qquad x = -y - z$

$\therefore (a,b,c) = (-y-z, y, z) = y(-1,1,0) + z(-1,0,1)$

$S = \{(-1,1,0) ; (-1,0,1)\} \qquad \dim S = 2$

$\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$

2	=	2		1		1
3	=	2		1		

$\forall v = (a,b,c) \in S \cap T$

$$\begin{cases} x - y + \lambda z = 0 \\ y + z = 0 \\ x + y + z = 0 \end{cases} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & \lambda \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & \lambda - 1 \\ 0 & 0 & \lambda + 1 \end{bmatrix}$$

$$\begin{cases} x + y + z = 0 \\ -2y + z(\lambda - 1) = 0 \\ z(\lambda + 1) = 0 \end{cases} \qquad y = \frac{1}{2} z (\lambda - 1) \qquad \begin{aligned} x &= -z - y \\ x &= -z - \frac{1}{2} z (\lambda - 1) \\ x &= z\left(-\frac{\lambda}{2} - \frac{1}{2}\right) \end{aligned}$$

$$= \left(z\left(\frac{-\lambda - 1}{2}\right), \frac{1}{2} z (\lambda - 1), z \right) = z \left(\frac{-\lambda - 1}{2}, \frac{\lambda - 1}{2}, 1 \right)$$

?

(118.)
$$\begin{cases} x - y + z = 0 \\ 2x + z - t = 0 \\ x - 3y + 2z - 2t = 0 \end{cases} \qquad \begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 0 & 1 & -1 \\ 1 & -3 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -2 & 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix} = 0 \qquad \begin{cases} x - y + z = 0 \\ 2y - z - t = 0 \end{cases} \qquad \begin{aligned} t &= 2y - z \\ x &= y - z \end{aligned}$$

$$(y - z, y, z, 2y - z) =$$

$$= y(1,1,0,2) + z(-1,0,1,-1)$$

$$W = \{(1,1,0,2), (-1,0,1,-1)\}$$

$$\dim W = 2$$

(119.) $S = \{1, x, x^2, x^3\}$

$T = \{u_1 = 1 - x^2; v_2 = 1 + 2x - x^2; v_3 = 1 + x + 2x^2 + 2x^3; u_4 = 1 + x - x^3\}$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$ $\therefore$ O conjunto é LI $\therefore$ forma uma base

120.)

122.) $W = \{(a,b,c,d) \mid a=b;\ c=d\}$

$\forall w = (a,b,c,d) = (b,b,d,d) = b(1,1,0,0) + d(0,0,1,1);\ b,d \in \mathbb{R}$

$W = \{(1,1,0,0);\ (0,0,1,1)\} \quad \dim = 2$

Lista de exercícios do Base
Álgebra A

1-)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

2-) $X = AB^t$, $A = (a_{ij})_{3\times3}$ tal que $a_{ij} = i + j$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ 1 & 4 & -1 \\ 2 & 1 & 6 \end{bmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad B^t = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 2 & -1 & -2 \\ 1 & 4 & -1 \\ 2 & 1 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & -1 \\ 1 & -2 & 4 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\det X \neq 0 \quad \therefore \quad \begin{vmatrix} 2 & 5 & -1 \\ 1 & -2 & 4 \\ 2 & 3 & 1 \end{vmatrix} = -4 + 40 - 3 - 4 - 24 - 5 = 0$$

$\therefore$ X não é inversível

3-) $X = AB^t$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ 1 & 4 & -1 \\ 2 & 1 & 8 \end{bmatrix} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad B^t = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$X = \begin{bmatrix} 2 & -1 & -2 \\ 1 & 4 & -1 \\ 2 & 1 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & -4 \\ 0 & -2 & -2 \\ 10 & 3 & 16 \end{bmatrix}$$

$$\det X = \begin{vmatrix} 0 & 5 & -4 \\ 0 & -2 & -2 \\ 10 & 3 & 16 \end{vmatrix} = -100 - 80 = -180 \neq 0 \quad \therefore \; X \text{ é inversível}$$

4-) Uma matriz é inversível quando $\det(A) \neq 0$

$$\det X = \begin{vmatrix} 1 & 1 & 2 \\ 4 & 5 & k^2 \\ -3 & 0 & k \end{vmatrix} = 5k - 3k^2 + 30 - 4k = 0$$

$$-3k^2 + k + 30 = 0$$

$$x = \frac{-1 \pm \sqrt{1+360}}{-6}$$

$$x_1 = \frac{-1+19}{-6} = -3$$

$$x_2 = \frac{-1-19}{-6} = \frac{20}{6} = \frac{10}{3}$$

A matri X é inversível quando $k \neq -3$ e $k \neq 10/3$

5-)

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$A = 3 \times \underbrace{2 \quad B = 2}_{=} \times 3$

Sim, pois o numero de colunas da matriz A é igual o número de linhas da matriz B. Com isto resulta numa matriz 3x3, pois é o número de linhas da matriz A versus o numero de colunas da matriz B.

$$C = A.B = \begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} -3 & 1 & -2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 5 & 0 \\ 9 & 4 & 7 \\ -1 & -2 & -1 \end{pmatrix}$$

6-)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3/2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2/3 & 1/2 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det A = 1 \quad \det B = 8 \quad \det C = 1$

$\therefore \det(ABC) = 8$

7-)

$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ $\quad \det A = 1$

$\operatorname{cof} A = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \quad (\operatorname{cof} A)^t = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

$$A^{-1} = \frac{1}{\det A} \cdot (\operatorname{cof} A)^t = \frac{1}{1} \cdot \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

ou

$A^{-1} = A I_n$

$$A^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

$a + 2b = 1$

$c + 2d = 0$

$a + 3b = 0$

$c + 3d = 1$

$$\begin{cases} a + 2b = 1 \\ a + 3b = 0 \end{cases} \sim \begin{cases} a + 2b = 1 \\ b = -1 \end{cases} \therefore a = 3$$

$$\begin{cases} c + 2d = 0 \\ c + 3d = 1 \end{cases} \sim \begin{cases} c + 2d = 0 \\ d = 1 \end{cases} \therefore c = -2$$

8-) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -2 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}$ $AX = B$

$3 \times 3 - 3 \times 2 = 3 \times 2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \therefore X = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{cases} a + b = -1 \\ d + e = -2 \\ a + c = 0 \\ d + f = 0 \\ -a + b + c = 2 \\ -d + e + f = 1 \end{cases}$$

$b = -1 - a$
$e = -2 - d$
$c = -a$
$f = -d$

$-a + b + c = 2$
$-a - 1 - a - a = 2$
$-3a = 3$
$a = -1$
$-d + e + f = 1$
$-d - 2 - d - d = 1$
$-3d = 3$
$d = -1$

$b = -1 + 1 = 0$
$c = -(-1) = 1$
$e = -2 + 1 = -1$
$f = -(-1) = 1$

9-) $A \cdot A^t = I_n$

$$\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

10-) $\begin{bmatrix} m-n & n+m \\ m+n & n-m \end{bmatrix}$ Para o matriz A seja ortogonal

$AA^t = I_n$ ou $A^t = A^{-1}$ e $\det A \pm 1$

$$\begin{bmatrix} m-n & n+m \\ m+n & n-m \end{bmatrix} \begin{bmatrix} m-n & m+n \\ n+m & n-m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \therefore \begin{bmatrix} 2(m^2+n^2) & 2(m^2-n^2) \\ 2(m^2-n^2) & 2(m^2+n^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2(m^2+n^2)=1 \quad m^2+n^2=1/2$

$2(m^2-n^2)=0 \quad m^2-n^2=0 \quad m^2=n^2$

$\therefore \; m^2+n^2=\frac{1}{2} \;\; e \;\; m^2=n^2$

11-) $M=\begin{pmatrix} 1/\sqrt{2} & 2/3 & 1/3\sqrt{2} \\ -1/\sqrt{2} & 2/3 & 1/3\sqrt{2} \\ a & b & c \end{pmatrix} \qquad M.M^t=I_n$

$$\begin{pmatrix} 1/\sqrt{2} & 2/3 & 1/3\sqrt{2} \\ -1/\sqrt{2} & 2/3 & 1/3\sqrt{2} \\ a & b & c \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & a \\ 2/3 & 2/\sqrt{3} & b \\ 1/3\sqrt{2} & 1/3\sqrt{2} & c \end{pmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$a/\sqrt{2}+2b/3+c/3\sqrt{2}=0 \Rightarrow \frac{a\sqrt{2}}{2}+\frac{2b}{3}+\frac{c\sqrt{2}}{6}=0 \Rightarrow$

$\Rightarrow 3a\sqrt{2}+4b+c\sqrt{2}=0$

$-a/\sqrt{2}+2b/3+c/3\sqrt{2}=0 \Rightarrow$

$\Rightarrow -3\sqrt{2}a+4b+c\sqrt{2}=0$

$\Rightarrow a^2+b^2+c^2=1$

$$\begin{cases} 3a\sqrt{2}+4b+c\sqrt{2}=0 \\ -3a\sqrt{2}+4b+c\sqrt{2}=0 \\ a^2+b^2+c^2=0 \end{cases}$$

$8b+2c\sqrt{2}=0 \quad \therefore a=0 \quad b=-\frac{1}{3} \quad c=\pm\frac{2\sqrt{3}}{3}$

$b=-\frac{1}{4}c\sqrt{2}$

$\to 0^2+b^2+c^2=0 \quad \therefore b^2+c^2=0$

$3a\sqrt{2}-c\sqrt{2}+c\sqrt{2}=0$

$3a\sqrt{2}=0 \quad \therefore a=0$

$$\begin{cases} 8b+2c\sqrt{2}=0 \\ b^2+c^2=1 \end{cases}$$

$\left(-\frac{1}{4}c\sqrt{2}\right)^2+c^2=1$

$\to c=\frac{1}{\sqrt{\frac{9}{8}}}=1:\frac{3}{\sqrt{8}}=\frac{\sqrt{8}}{3}=\frac{2\sqrt{2}}{3}$

$\frac{1}{8}c^2+c^2=1 \quad c^2\left(\frac{1}{8}+1\right)=1$

12-) $\begin{cases} x+2y+z=4 \\ 2x-3y+4z=3 \\ 3x-y+z=3 \end{cases} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & 4 & 3 \\ 3 & -1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & 2 & -5 \\ 0 & -7 & -2 & -9 \end{bmatrix} \to$

$\to \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & 2 & -5 \\ 0 & 0 & -4 & -4 \end{bmatrix} \sim \begin{cases} x+2y+z=4 \\ -7y+2z=-5 \\ -4z=-4 \end{cases}$ $\quad z=\frac{-4}{-4}=1$

$-7y+2z=-5$

$y=\frac{-5-2z}{-7} = \frac{-5-2}{-7} = 1 \quad x=4-2y-z$

$4-2-1=1$

$(x,y,z)=(1,1,1)$

13-) $\begin{cases} 3x+y+2z=21{,}5 \\ 8x+3y+5z=57{,}00 \\ x+y+z=10{,}00 \end{cases} \sim \begin{bmatrix} 1 & 1 & 1 & 10 \\ 3 & 1 & 2 & 21{,}5 \\ 8 & 3 & 5 & 57{,}5 \end{bmatrix} \to$

$\begin{bmatrix} 1 & 1 & 1 & 10 \\ 0 & -2 & -1 & -8{,}5 \\ 0 & -5 & -3 & -21{,}5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 10 \\ 0 & -2 & -1 & -8{,}5 \\ 0 & 0 & & \end{bmatrix}$

Exercícios Base

1-) $V = M_{2x2}(\mathbb{R})$ $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / a=d \text{ e } b=c \right\}$

$T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / a=c \text{ e } b=d \right.$

$S \cap T \Rightarrow$ $a-d=0$ $a=d$ $b=c$ $a=d$

$b-c=0$ $a=c$ $b=d$ $b=d$

$a-c=0$ $c=d$

$b-d=0$ $d=d$

$S \cap T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & d \\ d & d \end{bmatrix} = d \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\therefore S \cap T = \left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right]$ $\underline{\dim = 4}$

$\forall x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{S \cap T}{\in} = \begin{bmatrix} c & d \\ c & d \end{bmatrix} = c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\therefore T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $\underline{\dim = 2}$

c) $S+T = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ $\dim(S+T) =$

$\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$

$\dim(S+T) =$

2-) $S = [(1,0,2), (-2,3,1), (-1,3,3)]$

$T = \{(x,y,z) / x - 3y + 5z = 0\}$

a)

$\forall s = (x,y,z) \in S$

$(x,y,z) = a(1,0,2) + b(-2,3,1) + c(-1,3,3)$

$$\begin{cases} a - 2b - c = x \\ 3b + 3c = y \\ 2a + b + 3c = z \end{cases} \sim \begin{bmatrix} 1 & -2 & -1 & x \\ 0 & 3 & 3 & y \\ 2 & 1 & 3 & z \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & x \\ 0 & 3 & 3 & y \\ 0 & 5 & 5 & z-2x \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & -1 & x \\ 0 & 3 & 3 & y \\ 0 & 0 & 0 & -6x-5y+3z \end{bmatrix} \therefore \underline{-6x - 5y + 3z = 0}$$

b-) $x = 3y - 5z$

$\forall t = (x,y,z) \in T$

$\therefore (x,y,z) = (3y - 5z, y, z) = y(3,1,0) + z(-5,0,1)$

$$\therefore \begin{bmatrix} 3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 0 \\ 0 & 5 & 3 \end{bmatrix} \therefore \text{é } LI$$

logo $T = \{(3,1,0), (0,5,3)\}$ $\dim T = 2$

c-) $$\begin{cases} -6x - 5y + 3z = 0 \\ x - 3y + 5z = 0 \end{cases} \begin{bmatrix} -6 & -5 & 3 \\ 0 & -23 & 33 \end{bmatrix} \begin{cases} -6x - 5y + 3z = 0 \\ -23y + 33z = 0 \end{cases}$$

$y = \frac{33}{23} z \quad x = \frac{5y - 3z}{-6} = -\frac{16}{23} z \quad (x,y,z) = \left(\frac{-16}{23} z, \frac{33}{23} z, z\right)$

$= z\left(\frac{-16}{23}, \frac{33}{23}, 1\right) \therefore \left\{\left(\frac{-16}{23}, \frac{33}{23}, 1\right)\right\}$ $\dim . S \cap T = 1$

3) $U = [(1,0,0,0)]$ e $W = [(1,1,0,0),(0,1,1,0)]$

a) $U+W$ e justificar se a soma direta?

$U+W =$

4-) $a(-1,1,0,2) + b(1,3,1,0) = (0,0,0,0)$

$$\begin{pmatrix} -1 & 1 & x \\ 1 & 3 & y \\ 0 & 1 & z \\ 2 & 0 & t \end{pmatrix} \to \begin{bmatrix} -1 & 1 & x \\ 0 & 4 & x+y \\ 0 & 1 & z \\ 0 & 2 & 2x+t \end{bmatrix} \to \begin{bmatrix} -1 & 1 & x \\ 0 & 4 & x+y \\ 0 & 0 & x+y-4z \\ 0 & 0 & 2x+t-2z \end{bmatrix}$$

$U = \{(x,y,z,t) \mid x+y-4z=0 \text{ e } 2x-2z+t=0\}$

5-) $A = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+3y-z+t=0\}$
$B = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+4y-3z=0 \quad z-t=0\}$
$C = \{(1,0,4,-3),(1,1,2,-1),(2,1,6,-4)\}$

a) $x = -3y+z-t \qquad (-3y+z-t, y, z, t)$
$y(-3,1,0,0) + z(1,0,1,0) + t(1,0,0,1)$
$[(-3,1,0,0);(1,0,1,0);(1,0,0,1)]$

b)

$$\begin{cases} a + b + 2c = 0 \\ b + c = 0 \\ 4a + 2b + 6c = 0 \\ -3a - b - 4c = 0 \end{cases} \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 4 & 2 & 6 & 0 \\ -3 & -1 & -4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} \longrightarrow$$

30-) $L = \{(1 - x + 2x^2 - x^3,\ x + x^2 + x^3,\ 3 - x + 2x^2 + 2x^3;\ -1 + 2x + 5x^2 + mx^3\}$

$a(1-x+2x^2-x^3) + b(x+x^2+x^3) + c(3-x+2x^2+2x^3) + d(-1+2x+5x^2+mx^3)$

$$\begin{cases} a + 0 + 3c - d = 0 \\ -a + b - c + 2d = 0 \\ 2a + b + 2c + 5d = 0 \\ -a + b + 2c + m = 0 \end{cases} \sim \begin{pmatrix} -1 & 1 & -1 & 2 \\ 1 & 0 & 3 & -1 \\ 2 & 1 & 2 & 5 \\ -1 & 1 & 2 & m \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 3 & m-2 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} -1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 2m+5 \end{pmatrix} \qquad 2m + 5 = 0 \qquad m = \frac{-5}{2}$$

39-) $S = \left\{ \begin{bmatrix} x-y & y \\ -y & x+y \end{bmatrix} \Big/ x, y \in \mathbb{R} \right\}$

i) x $\quad x = 0,\ y = 0,\ z = 0$ $\quad \begin{bmatrix} 0-0 & 0 \\ -0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

ii) $\forall s_1 = \begin{bmatrix} x_1 & y_1 \\ z_1 & t_1 \end{bmatrix} \in S \quad \forall s_2 = \begin{bmatrix} x_2 & y_2 \\ z_2 & t_2 \end{bmatrix}$

$\therefore \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ z_1 + z_2 & t_1 + t_2 \end{bmatrix} \in S \quad$ iii) $\lambda \begin{bmatrix} x & y \\ z & t \end{bmatrix} \lambda = \begin{bmatrix} \lambda x & \lambda y \\ \lambda z & \lambda t \end{bmatrix}$

Prova P1B

1ª)

$$X^t (M^t + I)^{-1} = N$$

$$[X^t (M^t + I)^{-1}]^t = N^t$$

$$[(M^t + I)^{-1}]^t (X^t)^t = N^t$$

$$((M^t + I)^t)^{-1} X = N^t$$

$$(I + M)^{-1} X = N^t$$

$$(I + M)(I + M)^{-1} X = (I + M) N^t$$

$$X = (I + M) N^t$$

b) $N = [2\ 0\ 1]$ e $M = \begin{pmatrix} 3 & 2 & 5 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 5 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 5 \\ 1 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix} = I + M$$

$$N^t = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad (I+M) \cdot N^t = \begin{bmatrix} 4 & 2 & 5 \\ 1 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 7 \end{bmatrix}$$

2-) $S = \left\{ \begin{bmatrix} x-y & y \\ -y & x+y \end{bmatrix} ; x, y \in \mathbb{R} \right\}$

a) S é subspaços

i) $0 \in S$ pois $x=0 ; y=0 \in S$

$$\begin{bmatrix} 0-0 & 0 \\ -0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$$

ii) $\forall s_1 + \forall s_2 \in S$

$$s_1 = \begin{bmatrix} x_1-y_1 & y_1 \\ -y_1 & x_1+y_1 \end{bmatrix} ; x_1, y_1 \in \mathbb{R} \qquad S_2 = \begin{bmatrix} x_2-y_2 & y_2 \\ -y_2 & x_2+y_2 \end{bmatrix} ; x_2, y_2 \in \mathbb{R}$$

$$s_1 + s_2 = \begin{bmatrix} (x_1+x_2)-(y_1+y_2) & y_1+y_2 \\ -(y_1+y_2) & (x_1+x_2)+(y_1+y_2) \end{bmatrix} = \begin{bmatrix} x_1-y_1 & -y_1 \\ -y_1 & x_1+y_1 \end{bmatrix} + \begin{bmatrix} x_2-y_2 & -y_2 \\ -y_2 & x_2+y_2 \end{bmatrix}$$

$\therefore s_1 + s_2 \in S$

iii)

$\lambda \in \mathbb{R}$ e $\forall x, y \in \mathbb{R}$

$$\begin{bmatrix} (x_1-y_1)\lambda & y_1\lambda \\ -y_1\lambda & (x_1+y_1)\lambda \end{bmatrix} = \lambda \begin{bmatrix} x_1-y_1 & y_1 \\ -y_1 & x_1+y_1 \end{bmatrix} \in S$$

O conjunto S é subespaço

b) $x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}$ dim $S = 2$

c)

$$\begin{bmatrix} -3 & 4 \\ 5 & -4 \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$a - b = -3$

$\left.\begin{array}{l} b = 4 \\ -b = 5 \end{array}\right\} V$

$a + b = -4$

$\therefore$ A matriz A não pertence ao subespaço S

3ª Questão

$$\begin{cases} a + b + c = 24 \\ \frac{a}{2} + \frac{b}{3} + \frac{c}{4} = 7 \\ a + 3b + 4c = 84 \end{cases}$$

a = nº peças do 1º homem
b = " " " 2º "
c = " " " 3º "

$$\begin{bmatrix} 1 & 1 & 1 & 24 \\ 6 & 4 & 3 & 84 \\ 1 & 3 & 4 & 84 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 24 \\ 0 & -2 & -3 & -60 \\ 0 & 2 & 3 & 60 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 24 \\ 0 & -2 & -3 & -60 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) $\therefore \begin{cases} a + b + c = 24 \\ -2b - 3c = -60 \end{cases}$

b) $b = \frac{-60 + 3c}{-2} \Rightarrow b = 30 - \frac{3}{2}c$

$a + 30 - \frac{1}{2}c = 24 \quad a = -6 + \frac{1}{2}c \quad \therefore \left(-6 + \frac{1}{2}c \,;\, 30 - \frac{3}{2}c \,;\, c \right)$

$-6 + \frac{c}{2} > 0 \qquad \frac{c}{2} > 6 \qquad c > 12$

$\underline{12 < c < 20}$

$30 - \frac{3c}{2} > 0 \qquad \frac{3c}{2} < 30 \qquad c < 20$

Para $c = 16$

$a = -6 + \frac{16}{2} = 2$

$b = 30 - \frac{3.16}{2} - 6$

$c = 16$

4ª Questão

$L: \{(1,-1,0,m), (2,m^2,0,0), (m,0,0,1)\}$ é LD?

$$\begin{bmatrix} 1 & -1 & 0 & m \\ 2 & m^2 & 0 & 0 \\ m & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & m \\ 0 & m^2+2 & 0 & -2m \\ 0 & m & 0 & -m^2+1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & m \\ 0 & m^2+2 & 0 & -2m \\ 0 & 0 & 0 & -m^4+m^2+2 \end{bmatrix}$$

$-m^4 + m^2 + 2$

$m^2 = t$

$-t^2 + t + 2 = 0$

$-t^2 + t + 2 = 0$

$t = \frac{-1 \pm \sqrt{1+8}}{-2}$

$t_1 = -1$

$t_2 = 2$

$m = \pm\sqrt{2}$

Para L ser LD; $m = \pm\sqrt{2}$

b) $(-1,1,0,2)$ e $(1,3,1,0)$

$(x,y,z,t) = a(-1,1,0,2) + b(1,3,1,0)$, $a,b \in \mathbb{R}$

$$\begin{cases} -a+b = x \\ a+3b = y \\ b = z \\ 2a = t \end{cases} \sim \begin{bmatrix} -1 & 1 & x \\ 1 & 3 & y \\ 0 & 1 & z \\ 0 & 2 & t \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & x \\ 0 & 4 & x+y \\ 0 & 0 & x+y-4z \\ 0 & 0 & x+y-2t \end{bmatrix}$$

Equações homogeneas $\begin{cases} x+y-4z=0 \\ x+y-2t=0 \end{cases}$

Prova P1 13/10/09 a=pai b=mãe c=cintia d-Daniel

1ª) $\begin{cases} a+b+c+d=70 \\ a=3c+3d \\ b+10-20=2c+2d \\ c-4=b-a \end{cases}$ $\begin{cases} a+b+c+d=70 \\ a-3c-3d=0 \\ b-2c-2d=10 \\ a-b+c=4 \end{cases}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 70 \\ 1 & 0 & -3 & -3 & 0 \\ 0 & 1 & -2 & -2 & 10 \\ 1 & -1 & 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 70 \\ 0 & 1 & 4 & 4 & 70 \\ 0 & 0 & 6 & 6 & 60 \\ 0 & 2 & 0 & 1 & 64 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 70 \\ 0 & 1 & 4 & 4 & 70 \\ 0 & 0 & 6 & 6 & 60 \\ 0 & 0 & -8 & -7 & -76 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 70 \\ 0 & 1 & 4 & 4 & 70 \\ 0 & 0 & 6 & 6 & 60 \\ 0 & 0 & 0 & 6 & 24 \end{bmatrix}$$

/ /

$$\begin{cases} a + b + c + d = 70 \\ b + 4c + 4d = 70 \\ 6c + 6d = 60 \\ 6d = 24 \end{cases}$$

$d = \frac{24}{6} = 4$

$c = 10 - 4 = 6$

$b = 70 - 24 - 16 = 30$

$a = 70 - 30 - 6 - 4 = 30$

Pai = 30 Mãe = 30 Cintia = 6 Daniel 4

2º)

a) $X^t(A^t - I)^{-1} = B$

$[X^t(A^t - I)^{-1}]^t = B^t$

$[(A^t - I)^{-1}]^t (X^t)^t = B^t$

$[(A^t - I)^t]^{-1} X = B^t$

$(I - A)^{-1} X = B^t$

$(I - A)(I - A)^{-1} X = (I - A) B^t$

$X = (I - A) B^t$

b)

$$\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \\ x & y & z \end{pmatrix} \times \begin{pmatrix} \sqrt{2}/2 & 0 & x \\ \sqrt{2}/2 & 0 & y \\ 0 & 1 & z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{\sqrt{2}}{2}(x+y) \\ 0 & 1 & z \\ \frac{\sqrt{2}}{2}(x+y) & z & x^2+y^2+z^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$x = -y$ $\quad x^2 + y^2 + z^2 = 1$ $\quad x = \frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}, z = 0$

$z = 0$ $\quad x^2 + y^2 = 1$ $\quad$ ou

$x^2 + x^2 = 1$ $\quad x = \frac{-1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, z = 0$

$2x^2 = 1$ $\quad x = +\frac{1}{\sqrt{2}}$ $\quad y = -\frac{1}{\sqrt{2}}$

3ª Questão

$(0 + 0x + 0x^2 + 0x^3 + 0x^4) = a(3 + mx + 12x^2 - 4x^3 + x^4) + b(1 + x^2) + c(x + 3x^2 - x^3 + x^4) + d(x^2 + x^4)$

$(3a + b) + (am + c)x + (12a + b + 3c + d)x^2 + (-4a - c)x^3 + (a + c + d)x^4$

$$\begin{cases} 3a + b \\ am + 0 + c \\ 12a + b + 3c + d = 0 \\ -4a + 0 - c + 0 = 0 \\ a + 0 + c + d = 0 \end{cases} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ -4 & 0 & -1 & 0 & 0 \\ 12 & 1 & 3 & 1 & 0 \\ m & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ -4 & 0 & -1 & 0 & 0 \\ m & 0 & 1 & 0 & 0 \\ 12 & 1 & 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 1-m & -m & 0 \\ 12 & 1 & 3 & 1 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & m-4 & 0 \\ 12 & 1 & 3 & 1 & 0 \end{bmatrix}$$

$\therefore m - 4 = 0$

$\underline{m = 4}$

4ª Questão:

$U = \{(x,y,z,t) \mid x-y = z+t \ ; \ x-y-2t = 0\}$

$W = [(1,0,1,1),(0,1,-1,2)]$

a) $U \cap W$

$\forall w \in (x,y,z,t) = a(1,0,1,1) + b(0,1,-1,2)$

$$\begin{cases} a = x \\ b = y \\ a-b = z \\ a+2b = t \end{cases}$$

$x - y = z \Rightarrow x-y-z=0$

$x+2y = t \Rightarrow x+2y-t=0$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 4 & 1 & -1 \end{bmatrix}$$

$$\begin{cases} x-y-z-t=0 \\ x-y+0-2t=0 \\ x-y-z+0=0 \\ x+2y+0-t=0 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & -2 \\ 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix} \sim$$

$$\begin{cases} x-y-z-t=0 \\ 3y+z+0=0 \\ z-t=0 \\ t=0 \end{cases}$$

$(x,y,z,t) = (0,0,0,0)$

$\cap U W = \{(0,0,0,0)\}$

$$\begin{cases} x-y-z-t=0 \\ x-y-2t=0 \end{cases} \quad \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{cases} x-y-t-z=0 \\ t-z=0 \end{cases} \quad \begin{matrix} x=y+2z \\ t=z \end{matrix} \quad (y+2z, y, z, z) =$$

$$=[(1,1,0,0),(2,0,1,1)]$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$$V=\{(1,1,0,0),(0,-2,1,1)\}$$

$$U+W=[(1,1,0,0),(0,-2,1,1),(1,0,1,1),(0,1,-1,2)]$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & -1 & 5 \end{bmatrix}$$

$$U+W=\{(1,0,1,1),(0,1,-1,2),(0,0,0,-3),(0,0,-1,5)\}$$

Lista 1 - Exercício de Matrizes e Sistemas Lineares

1) $A = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$ $\qquad A^2 + 2A - 11I = ?$

$A^2 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & -1 \end{bmatrix}$

$2A = 2 \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix}$ $\qquad 11I = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$

$\begin{bmatrix} 9 & -4 \\ -8 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -11 \end{pmatrix}$

2-) $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ $\qquad AB = I$

$\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{cases} 4a + 3b = 1 & -4b + 3b = 1 \quad -b = 1 \quad b = -1 \\ 4c + 3d = 0 & 4 - 4d + 3d = 0 \quad 4 - d = 0 \quad d = 4 \\ a + b = 0 & a = -b \quad a = 1 \\ c + d = 1 & c = 1 - d \quad c = 1 - 4 = -3 \end{cases}$

$A^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$ $\qquad$ ou $\qquad A^{-1} = \frac{1}{d} \cdot (cof)^t$

3) 4)

$$\begin{cases} x+y+z=12 \\ 3x-y+2z=14 \\ 2x-2y+z=-3 \end{cases} \sim \begin{pmatrix} 1 & 1 & 1 & 12 \\ 3 & -1 & 2 & 14 \\ 2 & -2 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 12 \\ 0 & -4 & -1 & -22 \\ 0 & -4 & -1 & -27 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 1 & 1 & 12 \\ 0 & -4 & -1 & -22 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

S.I

5) $$\begin{cases} x+y-\lambda z=0 \\ x+\lambda y-z=0 \\ x+(1+\lambda)y+z=0 \end{cases} \quad \begin{pmatrix} 1 & 1 & -\lambda \\ 1 & \lambda & -1 \\ 1 & 1+\lambda & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -\lambda \\ 0 & \lambda-1 & -1+\lambda \\ 0 & \lambda & 1+\lambda \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -\lambda \\ 0 & \lambda-1 & \lambda-1 \\ 0 & 0 & \lambda-1 \end{pmatrix}$$

$\lambda-1=0$

$\lambda=1$

6) $$\begin{cases} x+y+z=k \\ kx+y+z=1 \\ x+y-z=k \end{cases} \quad \begin{pmatrix} 1 & 1 & 1 & k \\ k & 1 & 1 & 1 \\ 1 & 1 & -1 & k \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & k \\ 0 & 1-k & 1-k & 1-k^2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

SPI

7)

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B) $A(XA^{-1})^t = B$

$A^{-1}A(XA^{-1})^t = A^{-1}B$

$(XA^{-1})^t = A^{-1}B$

$[(XA^{-1})^t]^t = (A^{-1}B)^t$

$XA^{-1} = B^t(A^{-1})^t$

$AXA^{-1} = AB^t(A^{-1})^t$

$X = AB^t(A^{-1})^t$

-> $X^t(A^t+I)^{-1} = B$

$[X^t(A^t+I)^{-1}]^t = B^t$

$[(A^t+I)^{-1}]^t(X^t)^t = B^t$

$[(A^t+I)^t]^{-1}X = B^t$

$(A^t+I)^t[(A+I)^t]^{-1}X = (A^t+I)^t B^t$

$X = (A^t+I)^t B^t$

$X = (A+I)B^t$

-> $S = \{(1,-1,0,\alpha),(2,\alpha^2,0,0),(\alpha,0,0,1)\}$ ¿ LI?

$$\begin{bmatrix} 1 & -1 & 0 & \alpha \\ 2 & \alpha^2 & 0 & 0 \\ \alpha & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & \alpha \\ 0 & \alpha^2+2 & 0 & -2\alpha \\ 0 & \alpha & 0 & 1-\alpha^2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & \alpha \\ 0 & \alpha^2+2 & 0 & -2\alpha \\ 0 & 0 & 0 & -\alpha^4+\alpha^2+2 \end{bmatrix}$$

$-\alpha^4+\alpha^2+2=0$

$\alpha \neq \pm\sqrt{2}$

b) $(x,y,z,t) = a(1,-1,2,0) + b(1,0,3,1)$

$$\begin{cases} a+b=x \\ -a=y \\ 2a+3b=z \\ b=t \end{cases} \sim \begin{cases} a+b=x \\ b=x+y \\ b=z-2x \\ b=t \end{cases} \quad \begin{cases} 0=-3x-y-z \\ 0=x+y-t \end{cases}$$

Exercícios Extras

21-) $\begin{cases} x - 3y - z = 0 \\ 4x - 10y - 3z = 0 \\ mx - 5y - 4z = 0 \end{cases} \sim \begin{pmatrix} 1 & -3 & -1 \\ 4 & -10 & -3 \\ m & -5 & -4 \end{pmatrix} \sim$

$$\begin{pmatrix} 1 & -3 & -1 \\ 0 & 2 & 1 \\ 0 & -5+3m & -4+m \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -3-m \end{pmatrix}$$

$-3-m=0 \therefore m=-3$ SPI

$m \neq -3$ SPD

22-) $\begin{pmatrix} 1 & 0 & -1 & 2 & -3 \\ 1 & -2 & 5 & 2 & 7 \\ 2 & -1 & 1 & 2 & -1 \\ -1 & 1 & -3 & a & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 2 & -3 \\ 0 & -2 & 6 & 0 & 10 \\ 0 & -1 & 3 & -2 & 5 \\ 0 & 1 & -4 & a+2 & -2 \end{pmatrix} \sim$

$$\begin{pmatrix} 1 & 0 & -1 & 2 & -3 \\ 0 & -2 & 6 & 0 & 10 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & -2 & 2a+4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 2 & -3 \\ 0 & -2 & 6 & 0 & 10 \\ 0 & 0 & -2 & 2a+4 & 6 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

O conjunto será LI para qualquer valor de a

/ /

21) $(x,y,z,t)=a(1,-1,2,0)+b(1,1,3,-1)+c(0,1,1,2)$; $a,b,c \in \mathbb{R}$

$$\begin{cases} a+b = x \\ -a+b+c=y \\ 2a+3b+c=z \\ -b+2c=t \end{cases} \begin{bmatrix} 1 & 1 & 0 & x \\ -1 & 1 & 1 & y \\ 2 & 3 & 1 & z \\ 0 & -1 & 2 & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 2 & 1 & x+y \\ 0 & 1 & 1 & z-2x \\ 0 & -1 & 2 & t \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 2 & 1 & x+y \\ 0 & 0 & -1 & 5x+y-2z \\ 0 & 0 & 5 & x+y+2t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 2 & 1 & x+y \\ 0 & 0 & -1 & 5x+y-2z \\ 0 & 0 & 0 & 26x+6y-10z+2t \end{bmatrix}$$

$26x+6y-10z+2t=0$

$13x+3y-5z+t=0$

$v=(1,0,1,-1) \in [u_1,v_2,u_3]$ x

$13.1+3.0-5.1+1.-1 \neq 0$

25-) $U=\{(x,y,z) \mid x+2y-z=0\}$

$V=[(1,-1,2),(1,1,0),(2,0,2)]$

a) $U \cap V$

$(x,y,z)=a(1,-1,2)+b(1,1,0)+c(2,0,2)$

$$\begin{cases} a+b+2c=x \\ -a+b=y \\ 2a+2c=z \end{cases} \sim \begin{bmatrix} 1 & 1 & 2 & x \\ -1 & 1 & 0 & y \\ 2 & 0 & 2 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 2 & 2 & x+y \\ 0 & -2 & -2 & z-2x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 2 & 2 & x+y \\ 0 & 0 & 0 & -x+y+z \end{bmatrix}$$

$V=[(-1,0,0),(0,1,0),(0,0,1)]$

$-x+y+z=0$

$$\begin{cases} x+2y-z=0 \\ -x+y+z=0 \end{cases} \sim \begin{cases} x+2y-z=0 \\ 3y=0 \end{cases} \quad \begin{matrix} x-z=0 \\ y=0 \end{matrix}$$

$U \cap V = [(1,0,1)]$

$U \cap V = \{(1,0,1)\}$ $\dim = 1$

Livro 69)

$W = \{(x,y,z) \mid x+y+z=0\}$

i) para $\forall s_1 = (x_1, y_1, z_1) = (0,0,0)$ ∴ $0+0+0=0$ ∴ condição i verdadeira

ii) $\forall s_1 = (x_1, y_1, z_1) \in \mathbb{R}^3 \mid x_1+y_1+z_1=0$

$\forall s_2 = (x_2, y_2, z_2) \in \mathbb{R}^3 \mid x_2+y_2+z_2=0$

$s_1+s_2 = (x_1+x_2, y_1+y_2, z_1+z_2) \mid$

$(x_1+y_1+z_1)+(x_2+y_2+z_2)=0$ ∴ $u+v \in W$

iii) $\forall \lambda \in \mathbb{R}, \forall u \in W \mid x_1+y_1+z_1=0$

$\lambda u = (\lambda x_1 + \lambda y_1 + \lambda z_1) \mid \lambda \underbrace{(x_1+y_1+z_1)}_{=0} = 0$ ∴ $\lambda \in W$

$W = \{(x,y,z) \in \mathbb{R}^3 \mid y^2+z=0\}$

i) $0=(0,0,0) \in W$ pois $0^2+0=0$

ii) $\forall u = (x_1, y_1, z_1) \in \mathbb{R}^3$ pois $y_1^2+z_1=0$

$\forall v = (x_2, y_2, z_2) \in \mathbb{R}^3$ pois $y_2^2+z_2=0$

$u+v = (x_1+x_2, y_1+y_2, z_1+z_2) \mid$

$(y_1+y_2)^2+(z_1+z_2)=0$ $\quad y_1^2+2y_1y_2+y_2^2+z_1+z_2 \neq 0$

∴ $u+v \notin W$

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86) $U = [(1,0,0),(1,1,1)]$ $T = [(1,-1,2),(0,1,-1)]$

$(x,y,z) = a(1,0,0) + b(1,1,1) ; a,b \in \mathbb{R}$

$\begin{cases} a+b = x \\ b = y \\ b = z \end{cases}$ $\begin{cases} a+b = x \\ b = y \\ 0 = y-z \end{cases}$

$(x,y,z) = a'(1,-1,2) + b'(0,1,-1) ; a',b' \in \mathbb{R}$

$\begin{cases} a' = x \\ -a'+b' = y \\ 2a'-b' = z \end{cases} \sim \begin{cases} a' = x \\ b' = y+x \\ -b' = z-2x \end{cases} \sim \begin{cases} a' = x \\ b' = y+x \\ 0 = x+y-x \end{cases}$

87-) $(x,y,z) = a(1,-1,3) + b(0,1,-1) + c(2,3,1) ; a,b,c \in \mathbb{R}$

$\begin{cases} a+0+2c = x \\ -a+b+3c = y \\ 3a-b+c = z \end{cases} \sim \begin{cases} a+0+2c = x \\ 0 \ b+5c = x+y \\ 0 -b-5c = z-3x \end{cases} \sim \begin{cases} a+0+2c = x \\ 0+b+5c = x+y \\ 0+0+0 = -2x+y+z \end{cases}$

$\therefore$ Os vetores são coplanes, com isto eles gera o R^3

outra forma

$(1,-1,3) = a(0,1,-1) + b(2,3,1) ; a,b \in \mathbb{R}$

$\begin{cases} 2b = 1 \\ a+3b = -1 \\ -a+b = 3 \end{cases}$ $b = \frac{1}{2}$ $a = -1 - \frac{3}{2} = -\frac{5}{2}$

$-a+b = 3$

$-\left(-\frac{5}{2}\right) + \frac{1}{2} = \frac{6}{2} = 3$ ✓

$\therefore$ Como existe combinação linear de dois vetores com o terceiro, eles não geram R^3

88.) $u = \{(x,y,z) \mid x+2y+3z=0\}$
$w = \{(x,y,z) \mid 3x-y-z=0\}$

$U:$ $x = -2y-3z$
$(-2y-3z, y, z)$
$y(-2,1,0) + z(-3,0,3)$
$U = [(-2,1,0), (-3,0,3)]$

$W =$ $y = 3x - z$
$\forall (x,y,z) \in U /$
$(x,y,z) = (x, 3x-z, z)$
$= x(1,3,0) + z(0,-1,1)$
$W = [(1,3,0), (0,-1,1)]$

$U \cap W$:

$$\begin{cases} 3x - y - z = 0 \\ x + 2y + 3z = 0 \end{cases} \sim \begin{cases} 3x - y - z = 0 \\ 0 - 7y - 10z = 0 \end{cases}$$

$y = 3x - z$
$-7(3x - z) - 10z = 0$
$-21x + 7z - 10z = 0$
$-21x - 3z = 0$
$x = -\frac{1}{7} z$

$z = -7x$
$y = 3x - (-7x) = 10x$
$x = x$

$\therefore \forall (x,y,z) = (x, 10x, -7x)$
$= x(1, 10, -7)$
$U \cap W = [(1, 10, -7)]$

$U + W = [(-2,1,0), (-3,0,3), (1,3,0), (0,-1,1)]$

114-) $V = \mathbb{R}^4$ $U = \{(a,b,c,d) \mid a+c-d=0\}$

$W = \{(a,b,c,d) \mid a+d=0,\ c-2b=0\}$

$$\begin{cases} a+0+0+d=0 & a=-d \\ 0-2b+c+0=0 & c=2b \end{cases}$$

$\forall\ (a,b,c,d) \in W$

$(a,b,c,d) = (-d, b, 2b, d)$

$= d(-1,0,0,1) + b(0,1,2,0)$

$$\begin{cases} -a+\quad +d=0 \\ b+2c \quad =0 \end{cases}$$

$\therefore\ W = \{(-1,0,0,1),(0,1,2,0)\}$

$\dim = 2$

$U \cap W$: $\begin{cases} a+c-d=0 \\ a+\quad d=0 \\ c-2b=0 \end{cases}$ $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & -2 & 1 & 0 \end{bmatrix}$

$$\begin{cases} a+\quad +c-d=0 & a=d-c=d-2d=-d \\ -2b+c+0=0 & -2b=-c \quad 2b=2d \quad b=d \\ -c+2d=0 & c=2d \end{cases}$$

$\forall (a,b,c,d) = (-d, d, 2d, d)$

$= d(-1,1,2,1)$

$U \cap W = [(-1,1,2,1)]$ $\dim(U \cap W) = 1$

Exercícios "TREINO" Dependência linear

➡ $v_1 = (1,2,0)$ $v_2 = (-1,0,1)$ $v_3 = (1,2,3)$ (Dependência linear dos vetores)

Pela definição:

$a(1,2,0) + b(-1,0,1) + c(1,2,3) = (0,0,0)$

$$\begin{cases} a - b + c = 0 \\ 2a + 2c = 0 \\ b + 3c = 0 \end{cases} \quad \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix} \therefore \text{é LI}$$

ou

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \therefore \text{É LI}$$

Obs.: Para determinar dependência linear dos vetores não importa a posição $m \times n$ ou $n \times m$, pois o resultado é o mesmo.

- Linearmente Dependente é quando uma linha é nula
- " Independente é quando não existe linha

➡ $\{v_1 = (1,-1,0,2,1)$ $v_2 = (2,0,0,1,2)$ $v_3 = (0,1,-1,0,0)\}$ $V = \mathbb{R}^5$

$$\begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 2 & 1 & 2 \end{pmatrix} \therefore \text{O conjunto } \{v_1, v_2, v_3\} \text{ é LI}$$

(Depindância linear dos polinômios)

⇒ $p_1(t) = 1$; $p_2(t) = 2+t$; $p_3(t) = t + 2t^2$;

$a + b(2+t) + c(t+2t^2) = 0 + 0t + 0t^2$

$$\begin{cases} a + 2b = 0 \\ b + c = 0 \\ 2c = 0 \end{cases} \qquad \begin{aligned} a &= 0 \\ b &= 0 \\ c &= 0 \end{aligned}$$

Logo, $\{p_1(t), p_2(t), p_3(t)\}$ é LI

- Conjunto $\{v_1, v_2, \dots, v_3\}$ é LD se $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$
  Pelo menos um dos escolaus não nulo
- Conjunto $\{v_1, v_2, \dots, v_3\}$ é LI se $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$, se somente $a_1 = a_2 = \dots = a_n = 0$

⇒ $S = \{1-x,\ 1-x-x^2,\ 2x+x^2\}$

$a(1-x) + b(1-x-x^2) + c(2x+x^2) = 0 + 0x + 0x^2$

$(a+b) + x(-a-b+2c) + x^2(-b+c) = 0 + 0x + 0x^2$

$$\begin{cases} a + b = 0 \\ a - b - 2c = 0 \\ -b + c = 0 \end{cases} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Logo, S é LI

$u = 2 - 4t + t^2 - t^3$

$v = -1 + t - t^3$

$w = 1 + 2t + t^2 + 2t^3$

$a(2-4t+t^2-t^3) + b(-1+t-t^3) + c(1+2t+t^2+2t^3) = 0 + 0t + 0t^2 + 0t^3$

$(2a-b+c) + t(-4a+b+2c) + t^2(a+c) + t^3(-a-b+2c) = 0 + 0t + 0t^2 + 0t^3$

$$\begin{cases} 2a - b + c = 0 \\ -4a + b + 2c = 0 \\ a + 0 + c = 0 \\ -a - b + 2c = 0 \end{cases} \quad \begin{bmatrix} 2 & -1 & 1 \\ -4 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & 4 \\ 0 & -1 & -1 \\ 0 & -3 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \therefore \underline{(\text{Dúvida})}$$

$p_1 = 3 + mx + 12x^2 - 4x^3 + x^4$

$p_2 = 1 + x^2$

$p_3 = x + 3x^2 - x^3 + x^4$

$p_4 = x^2 + x^4$

$a(3 + mx + 12x^2 - 4x^3 + x^4) + b(1 + x^2) + c(x + 3x^2 - x^3 + x^4) + d(x^2 + x^4) = 0 + 0x + 0x^2 + 0x^3$

$(3a+b) + x(am+c) + x^2(12a+b+3c+d) + x^3(-4a-c) + x^4(a+c+d) = 0 + 0x$

$$\begin{cases} 3a + b = 0 \\ am + \quad + c = 0 \\ 12a + b + 3c + d = 0 \\ -4a \quad - c = 0 \\ a + \quad + c + d = 0 \end{cases} \quad \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ -4 & 0 & -1 & 0 \\ m & 0 & 1 & 0 \\ 12 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 \\ 0 & 4 & -3 & 0 \\ 0 & -m & 3 & 0 \\ 0 & -9 & 9 & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 \\ 0 & 4 & -3 & 0 \\ 0 & -m & 3 & 0 \\ 0 & -9 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 9 & 12 \\ 0 & 0 & 3-3m & -3m \\ 0 & 0 & -18 & -24 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 9 & 12 \\ 0 & 0 & 0 & -36+9m \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-36+9m=0 \quad \therefore \underline{m=4}$

* $\{f(t)=1;\ g(t)=e^t;\ h(t)=e^{2t}\}$

$a + be^t + ce^{2t} = 0 + 0^t + 0^{2t}$

Admitindo $t=0$ | Admitindo $t=1$ | Admitindo $t=2$

$a+be^0+ce^{2.0}=0$ | $a+be^1+ce^{2.1}=0$ | $a+be^2+ce^{2.2}=0$

$a+b+c=0$ | $a+eb+e^2c=0$ | $a+be^2+ce^4=0$

$$\begin{cases} a+b+c=0 \\ a+be+ce^2=0 \\ a+be^2+ce^2=0 \end{cases} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e & e^2 \\ 1 & e^2 & e^4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & e-1 & e^2-1 \\ 0 & e^2-1 & e^4-1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & e-1 & e^2-1 \\ 0 & 0 & e^5-e^4+2e^2-e \end{bmatrix}$$

$\neq 0$

$\therefore$ LI

Prova P_1 - 2006

1ª Questão:

$AA^t = I \quad A^{-1}=A^t$

sendo $A= \begin{bmatrix} a & c \\ b & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad a^2+c^2=1 \quad ab+cd=0$

$b^2+d^2=1 \quad ab+cd=0$

B) $\begin{pmatrix} p-q & q+p \\ p+q & q-p \end{pmatrix} \times \begin{pmatrix} p-q & p+q \\ q+p & q-p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(p-q)^2 + (q+p)^2 = 1 \Rightarrow p^2 - \cancel{2pq} + q^2 + q^2 + \cancel{2pq} + p^2 = 1$

$(p-q)(p+q) + (q+p)(q-p) = 0 \Rightarrow \cancel{p^2} - \cancel{q^2} + \cancel{q^2} - \cancel{p^2} = 0$

$(p+q)(p-q) + (q-p)(q+p) = 0 \Rightarrow \cancel{p^2} - q^2 + \cancel{q^2} - q^2 = 0.$

$(p+q)^2 + (q-p)^2 = 1 \Rightarrow p^2 + \cancel{2pq} + q^2 + q^2 - \cancel{2pq} + p^2 = 1$

$2p^2 + 2q^2 = 1 \qquad p = \sqrt{\frac{1 - 2q^2}{2}}$

$2p^2 + 2q^2 = 1$

Provo P_1 2009

1-) $(AB - BA)^t = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^t$

$B^t A^t - A^t B^t = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^t$

$-(A^t B^t - B^t A^t) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

$A^t B^t - B^t A^t = -\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$

2-) $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 & 3 \\ x & -1 & -1 \\ y & z & -1 \end{bmatrix} \quad B^{-1} = A$

$B^{-1} = A$

$BB^{-1} = BA$

$I = BA$

$\begin{bmatrix} 4 & 3 & 3 \\ x & -1 & -1 \\ y & z & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 & 3 \\ x & -1 & -1 \\ y & z & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x + 1 = 0 \quad x = -1$

$\begin{cases} y + 2z = -3 \\ 3y + 5z = -9 \end{cases} \sim \begin{cases} y + 2z = -3 \\ -z = 0 \end{cases} \quad y = -3$

$\therefore x = -1 \quad y = -3 \quad z = 0$

4-) $S = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$
$T = \{(x,y,z) \in \mathbb{R}^3 \mid x-2y+\lambda z = 0,\ y+z=0\}$

a) Defina Soma Direta de 2 subespaços
→ A soma direta de 2 subespaços existirá se apenas a intersecção deles formar um valor nulo e a soma deles pertencer ao conjunto.

$S \cap T: \begin{cases} x+y+z=0 \\ x-2y+\lambda z=0 \\ y+z=0 \end{cases} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & \lambda \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & \lambda-1 \\ 0 & 0 & -2-\lambda \end{bmatrix}$

$-2-\lambda = 0$
$\lambda = -2$

3-) $G = \{(x,y,z,t) \mid x-y+z+t=0\}$ e $H = [(1,2,0,1),(2,3,0,3),(3,2,1,2)]$

a) Geradores de G
$\forall g = (x,y,z,t) \in G$
$\therefore (x,y,z,t) = (y-z-t, y, z, t) = y(1,1,0,0) + z(-1,0,1,0) + t(-1,0,0,1)$
$G = [(1,1,0,0), (-1,0,1,0), (-1,0,0,1)]$

b) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 0 & 3 \\ 3 & 2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & -4 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -5 \end{bmatrix} \therefore LI$

c) $G \cap H$

$(x,y,z,t) = a(1,2,0,1) + b(2,3,0,3) + c(3,2,1,2)\ ;\ a,b,c, \in \mathbb{R}$

$$\begin{cases} a + 2b + 3c = x \\ 2a + 3b + 2c = y \\ \quad\quad c = z \\ a + 3b + 2c = t \end{cases} \begin{bmatrix} 1 & 2 & 3 & x \\ 2 & 3 & 2 & y \\ 1 & 3 & 2 & t \\ 0 & 0 & 1 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & x \\ 0 & -1 & -4 & y-2x \\ 0 & 1 & -1 & t-x \\ 0 & 0 & 1 & z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & x \\ 0 & -1 & -4 & y-2x \\ 0 & 0 & -5 & -3x+y+t \\ 0 & 0 & 1 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & x \\ 0 & -1 & -4 & y-2x \\ 0 & 0 & -5 & -3x+y+t \\ 0 & 0 & 0 & -3x+y+5z+t \end{bmatrix}$$

$$\begin{cases} -3x + y + 5z + t = 0 \\ x - y + z + t = 0 \end{cases} \sim \begin{cases} -3x + y + 5z + t = 0 \\ -2y + 8z + 4t = 0 \end{cases}$$

$y = 4z + 2t \qquad x = \dfrac{4z + 2t + 5z + t}{-3} = -3z - t$

$(x,y,z,t) = (-3z - t,\ 4z + 2t,\ z,\ t)$
$= z(-3,4,1,0) + t(-1,2,0,1)$

$\begin{bmatrix} -3 & 4 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -3 & 4 & 1 & 0 \\ 0 & -2 & 1 & -3 \end{bmatrix}$ $\quad G \cap H = \{(-3,4,1,0), (0,-2,1,-3)\}$

$\dim = 2$

Exercícios (Victor - Erivelton)

107) $\{1-x,\ 1-x-x^2,\ 2x+x^2\}$

$a(1-x)+b(1-x-x^2)+c(2x+x^2) = 0+0x+0x^2$

$(a+b)+x(-a-b+2c)+x^2(-b+c) = 0+0x+0x^2$

$$\begin{cases} a+b = 0 \\ -a-b+2c = 0 \\ -b+c = 0 \end{cases} \sim \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} \therefore \text{O conjunto é LI}$$

113) $S = [(1,-1,2);(0,1,-1)]$ $T = [(-1,3,1);(2,1,0)]$

$\forall s\ (x,y,z) = a(1,-1,2)+b(0,1,-1);\ a,b \in \mathbb{R}$

$$\begin{cases} a = x \\ -a+b = y \quad b = y+a \Rightarrow b = y+x \\ 2a-b = z \quad 2.x-y-x = z \ \therefore\ x-y-z=0 \end{cases}$$

$\forall x = (x,y,z) \in T$

$\therefore (x,y,z) = a(-1,3,1)+b(2,1,0);\ a,b \in \mathbb{R}$

$$\begin{cases} -a+2b = x \quad -z+2y-6z-x=0 \ \therefore\ -x+2y-7z=0 \\ 3a+b = y \quad b = y-3a \quad b = y-3z \\ a = z \quad a = z \end{cases}$$

$$\begin{cases} x-y-z=0 \\ -x+2y-7z=0 \end{cases} \sim \begin{cases} x-y-z=0 \\ y-8z=0 \end{cases} \quad x = z+y = 9z \quad y = 8z$$

$(x,y,z) = (9z, 8z, z) = z(9,8,1)$ $\therefore$ $S \cap T = \{(9,8,1)\}$

Nome: Erivelton Gualter dos Santos 11.210.368-4

1º semestre de 2011 2º ciclo Engenharia Básica

Algebra linear (Parte II)

Recordando:

$\mathcal{U}, V$ e $\mathcal{V}$. sobre $\mathbb{R}$

$F: \mathcal{U} \to V$ é uma T.L. se

i) $F(u+v) = F(u) + F(v), \forall u, v, \in \mathcal{U}$

ii) $F(\lambda . u) = \lambda F(u), \forall \lambda \in \mathbb{R}, \forall u \in \mathcal{U}$

1) $Ker(F) = \{u \in \mathcal{U} / F(u) = 0_V\} \subset \mathcal{U}$

2) Se F é linear, $F(0_u) = 0_V$ se.

Obs.: Qdo $\mathcal{U} = V$ e $F: \mathcal{U} \to V$ é uma T.L, F é dito um Operador Linear

Imagem de uma TL

Seja $F: \mathcal{U} \to V$ linear

Def.: $Im(F) = \{v \in V / \exists u \in \mathcal{U}$ com $F(u) = v\}$

Ex 1: $F: \overset{\mathcal{U}}{\underset{\shortparallel}{\mathbb{R}}} \to \overset{V}{\underset{\shortparallel}{\mathbb{R}}}$

$x \longmapsto F(x) = x^2$

$F(x_1) = x_1^2$

$F(x_2) = x_2^2$

$F(x_1 + x_2) = (x_1 + x_2)^2$

$F(x_1) + F(x_2) = x_1^2 + x_2^2$ $\therefore F$ não é linear

Ex 2: $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$(x, y, z) \longmapsto F(x, y, z) = (x, y)$

i) $F((x_1, y_1, z_1) + (x_2, y_2, z_2)) = F(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (x_1 + x_2, y_1 + y_2) = (x_1, y_1) + (x_2, y_2)$

$= F(x_1, y_1, z_1) + F(x_2, y_2, z_2)$

ii) $F(\lambda(x, y, z)) = F(\lambda x, \lambda y, \lambda z) = (\lambda x, \lambda y) = \lambda(x, y)$

$= \lambda F(x, y, z)$ $\therefore F$ é Linear

iii) $Ker(F) = \{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = (x,y) = (0,0)\}$
$= \{(0,0,z) \mid z \in \mathbb{R}\}$

a) $Ker(F) = [(0,0,1)]$ $\therefore$ $B = ((0,0,1))$ é base para o $Ker(F)$
b) $\{(0,0,1)\}$ é li

$(2,1) \in Im(F)$?
$\hookrightarrow \in \mathbb{R}^2 = V$

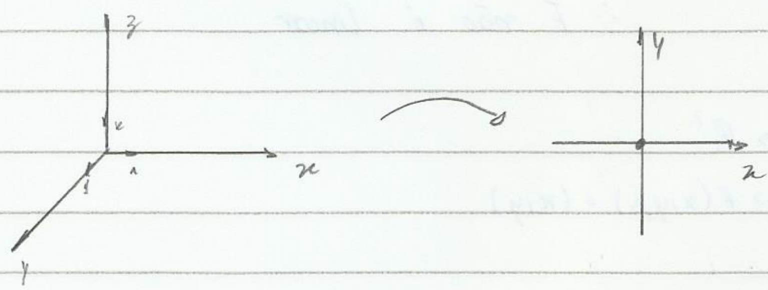
$(2,1,z) \in \mathbb{R}^3 \mid F(2,1,z) = (2,1)$

iv) $Im(F) = \{(u,v) \in \mathbb{R}^2 \mid \exists\, (x,y,z) \in \mathbb{R}^3$ com $F(x,y,z) = (x,y) = (u,v)\}$

$(u,v) = (x,y) = x(1,0) + y(0,1)$

a) $Im(F) = [(1,0),(0,1)]$
b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\therefore$ $\{(1,0);(0,1)\}$ é li

$C = ((1,0);(0,1))$ é base de $Im(F)$ $\therefore$ $Im(F) = \mathbb{R}^2$



Proposição: Seja $F: U \to V$ uma T.L
Então, $Im(F) \subset V$ é subespaço vetorial

Prova: (exercício)

Teorema 1 Se $F: U \to V$ é uma transformação linear, então:

$$\boxed{\dim U = \dim(\ker(F)) + \dim(Im(F))}$$

Ex 2: $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$(x,y) \longmapsto F(x,y) = (x+y, x-y, 2x+y)$

a) F é linear
b) $\ker(F) = ?$
$F(x,y) = (x+y, x-y, 2x+y) = (0,0,0)$

$$\begin{cases} x+y=0 & x+x=0 & 2x=0 & x=0 \\ x-y=0 & x=y & y=0 & \\ 2x+y=0 & 2 \cdot 0+0=0 & & \end{cases}$$

$\ker(F) = \{(0,0)\}$ $\therefore$ $\dim \ker(F) = 0$

c) $= x(1,1,2) + y(1,-1,1)$
i $= [(1,1,2), (1,-1,1)] = Im(F)$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

Logo, $Im(1) \subsetneq \mathbb{R}^3$

ii) $Im(F) = \{(1,1,2)(0,-2,-1)\}$ $\dim Im(F) = 2$

Teorema 2: Sejam U e V espaços vetoriais com dimensão n e m, respectiv.
Se $B = (u_1, u_2, \dots, u_n)$ é uma base de U e $v_1, v_2, \dots, v_n$ são vetores de V, então existe uma única transf. linear F tal que $F(u_1) = v_1$, $F(u_2) = v_2$, ..., $F(u_n) = v_n$

Ex.1: Sabendo-se que $F: \mathbb{R}^2 \to \mathbb{R}^3$ é tal que: $F(1,1) = (1,0,-1)$
$F(0,1) = (2,3,4)$,

determine $F(x,y)$.

$B = ((1,1),(0,1))$ é base do $\mathbb{R}^2$; assim, $\forall (x,y) \in \mathbb{R}^2$, existem n.os reais α e β tais que: $(x,y) = \alpha(1,1) + \beta(0,1)$

$$
\begin{aligned}
F(x,y) &= F(\alpha(1,1) + \beta(0,1)) \\
&= F(\alpha(1,1)) + F(\beta(0,1)) \\
&= \alpha F(1,1) + \beta F(0,1) \\
&= \alpha(1,0,-1) + \beta(2,3,4) =
\end{aligned}
$$

$(x,y) = (\alpha, \alpha + \beta)$

$x = \alpha$

$y = \alpha + \beta \Rightarrow \beta = y - \alpha = y - x$

$$
\begin{aligned}
F(x,y) &= \alpha(1,0,-1) + \beta(2,3,4) \\
&= x(1,0,-1) + (y-x)(2,3,4) \\
&= (x,0,-x) + (2(y-x), 3(y-x), 4(y-x))
\end{aligned}
$$

$F(x,y) = (2y - x, 3y - 3x, 4y - 5x)$

Conferindo... $F(1,1) = (1,0,-1)$
$F(0,1) = (2,3,4)$

Transformação Linear

Resumo

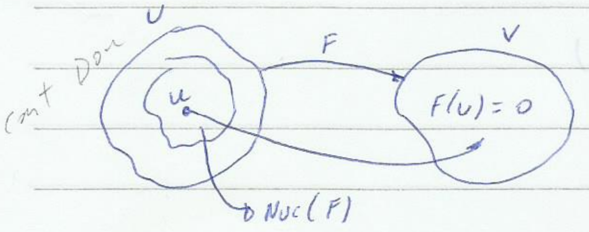
$F: U \to V$ Linear $\iff \begin{cases} i) \ F(u_1 + u_2) = F(u_1) + F(u_2) \\ ii) \ F(\lambda u) = \lambda . F(u) \end{cases}$

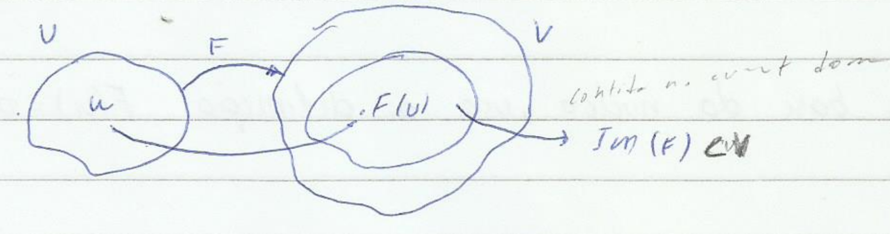
núcleo (ou kernel) de uma T.L.

$Nuc(F) = \{ u \in U \mid F(u) = 0 \}$

Imagem de uma TL

$IM(F) = \{ F(u) \in V \mid u \in U \}$





Teorema do Núcleo e da Imagem

(Resumo)

$$\boxed{\dim U = \dim Nuc(F) + \dim IM(F)}$$

1) Transformação linear injetora

proposição

$F: U \to V$ Linear é <u>Injetora</u> $\iff Nuc(F) = \{0\}$

/ /

2) Transformação linear sobrejetora

$F: U \to V$ é sobrejetora quando $Im(F) = V$

Se $F: U \to V$ for injetora e sobrejetora então F é um <u>isomorfismo</u>
Se $F: U \to U$ for injetora e sobrejetora então F é um <u>automorfismo</u>
Neste caso, existe F^{-1} que é inversa de F.

Exercício
Dada a TL $F: \mathbb{R}^3 \to \mathbb{R}^2 \mid F(x-y-z;\ 2x-3y+2z)$
Determinar:

a) Uma base do Nuc (F) e a dim Nu(F)
b) " " da Im (F) " " " Im (F)
Justificar se
i) F é injetora
ii) F é sobrejetora

a) Par obter a base do nucleo, usa a definição $F(u) = 0$

$F(x,y,z) = (0,0)$

$$\begin{cases} x - y - z = 0 \\ 2x - 3y + 2z = 0 \end{cases} \sim \begin{cases} x - y - z = 0 \\ -y + 4z = 0 \end{cases} \qquad \begin{aligned} y &= 4z \\ x &= z + y = 5z \end{aligned}$$

$\therefore\ y = 4z$ e $x = 5z$

$(x,y,z) \in Nuc(F) \mid$
$(x,y,z) = (5z, 4z, z) = z(5,4,1)$
Base $Nuc(F) = \{(5,4,1)\}$ dim $Nu(F) = 1$

→ Como $Nuc(F) \neq \{0\}$
F não é injetora

b) $Im(F) = \{F(u) \in V \mid u \in U\}$ $F: \mathbb{R}^3 \to \mathbb{R}^2 \mid F(x-y-z; 2x-3y+2z)$

Base da $Im(F)$

$$
\begin{aligned}
F(1,0,0) &= (1,2) \\
F(0,1,0) &= (-1,-3) \\
F(0,0,1) &= (-1,2)
\end{aligned}
\rightarrow
\begin{bmatrix} 1 & 2 \\ -1 & -3 \\ -1 & 2 \end{bmatrix}
\sim
\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 4 \end{bmatrix}
\sim
\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}
$$

Base $Im(F) = \{(1,2), (0,-1)\}$

$\dim Im(F) = 2$

F é sobrejetora pois $Im(F) = V$, isto é,

$\dim Im(F) = 2 = \dim V$

$\therefore$ F não é isomorfismo

2) Dada a TL $F: \mathbb{R}^3 \to \mathbb{R}^3$, $F(x,y,z) = (x, x-y, x-y-z)$

Determinar:

a) uma base do $Nuc(F)$ e $\dim Nuc(F)$

b) " " da $Im(F)$ " $Im(F)$

Justificar se:

i) F é injetora

ii) F é sobrejetora

a) $F(u) = 0$ $(x,y,z) \in Nuc(F) \mid$

$F(x,y,z) = (0,0)$ $(x,y,z) = (0,0,0)$

$$
\begin{cases}
x = 0 \\
x - y = 0 & y = 0 \\
x - y - z = 0 & z = 0
\end{cases}
$$

$F(u)$ é injetora pois $Nuc(F) = 0$

Base $Nuc(F) = \emptyset$

$\dim Nuc(F) = 0$

/ /

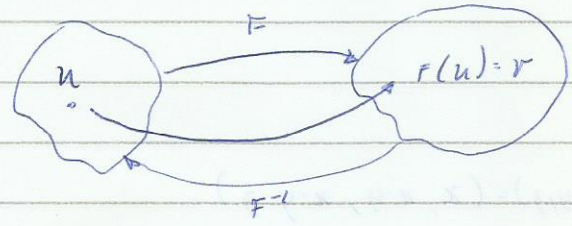
b) Base da imagem

$F(1,0,0) = (1,1,1)$
$F(0,1,0) = (0,-1,-1)$
$F(0,0,1) = (0,0,-1)$

Base $Im(F) = \{(1,1,1),(0,-1,-1),(0,0,-1)\}$
$\dim Im F = 3$

$\dim Im(F) = 3 = \dim V$. F é sobrejetora $\Big\{$ F é automorfismo

Cálculo da ~~~~inversa : F^{-1}



$F(u) = v$; $F^{-1}(v) = u$

$F^{-1}(x,y,z) = (a,b,c)$
$\underbrace{F(a,b,c)} = (x,y,z)$
$(a, a-b, a-b-c) = (x,y,z)$

$$\begin{cases} a = x \\ a-b = y \Rightarrow b = x-y \\ a-b-c = z \Rightarrow c = y-z \end{cases}$$

$\therefore F^{-1}(x,y,z) = (x, x-y, y-z)$

U, V e v sobre $\mathbb{R}$

$F: U \to V$ é linear-se:

transformação linear é o conjunto que respeita

i) $F(u+v) = F(u) + F(v), \forall u, v \in U$

ii) $F(\lambda u) = \lambda F(u), \forall \lambda \in \mathbb{R}, \forall u \in U$

Sejam U, V e v.

$L(U,V)$ conjunto de todas as transformações lineares de U em V

$L(U,V) = \{F: U \to V \mid F \text{ é linear}\}$

Ex: $U = \mathbb{R}^2$; $V = M_2(\mathbb{R})$

$L(\mathbb{R}^2, M_2(\mathbb{R}))$: conjunto de todas t.L de $\mathbb{R}^2$ em $M_2(\mathbb{R})$

Obs.: Quando $U = V$, indicamos por $L(U)$.

Espaço vetorial, qualquer conjunto de vetores, que satisfaça as 4 propriedades da adição e multiplicação

Operações com Transformações Lineares

1. Adição

Se eu conheço a base, eu conheço todo elemento do conjunto

$F, G \in L(U,V)$

Def: $F+G: U \to V$

$u \longmapsto (F+G)(x) = F(u) + G(u)$

Exercício Mostre que $F+G$ é linear (e, portanto, $F+G \in L(U,V)$).

A_1: $F+G = G+F$

A_2: $(F+G)+H = F+(G+H)$

A_3: $\exists \vec{0}: U \to V$

$u \longmapsto \vec{0}(u) = 0_V, \forall u \in U$

/ /

(transf. nula) tal que

$F + \bar{0} = F, \forall F \in L(U,V)$

A_4: $\forall F \in L(U,V)$, existe $-F \in L(U,V)$ tal que $F + (-F) = \bar{0}$

2 Multiplicação por Escalar

Def: $\lambda F: U \longrightarrow V$

$u \longmapsto (\lambda F)(u) = \lambda F(u)$

$\lambda \in \mathbb{R}, F \in L(U,V)$

Exercício: λF é linear

$(\lambda F \in L(U,V))$

Propriedades

M_1: $(\alpha, \beta) F = \alpha(\beta F), \forall \alpha, \beta \in \mathbb{R} \quad \forall F \in L(U,V)$

M_2: $(\alpha + \beta) F = \alpha F + \beta F, \forall \alpha, \beta \in \mathbb{R} \quad \forall F \in L(U,V)$

M_3: $\alpha(F + G) = \alpha F + \alpha G, \forall \alpha \in \mathbb{R}, \quad \forall F, G \in L(U,V)$

M_4: $1F = F, \forall F \in L(U,V)$

Conclusão

$(L(U,V), +, \cdot)$ é espaço vetorial sobre $\mathbb{R}$

3. Composição de Transf. Lineares

Def: Sejam $\mathcal{U}, V$ e W e.v. e $F \in L(U,V)$; $G \in L(V,W)$

$$\boxed{\begin{array}{l} G \circ F : \mathcal{U} \longrightarrow W \\ \quad u \longmapsto (G \circ F)(u) \overset{def}{=} G(F(u)) \end{array}}$$

$$\begin{tikzcd} \mathcal{U} \arrow[r, "F"] & V \arrow[r, "G"] & W \\ u \arrow[rr, bend right, "(G \circ F)(u)"'] & v = F(u) & w = G(F(u)) \end{tikzcd}$$

Exercício: $G \circ F$ é linear:

De fato i) $(G \circ F)(u_1 + u_2) = G(F(u_1 + u_2))$

$\overset{F \text{ é linear}}{=} G(F(u_1) + F(u_2))$

$\overset{G \text{ é linear}}{=} G(F(u_1)) + G(F(u_2))$

$= (G \circ F)(u_1) + (G \circ F)(u_2)$, $\forall u_1, u_2 \in \mathcal{U}$

ii) $(G \circ F)(\lambda u) \overset{def}{=} G(F(\lambda u))$

$\overset{F \text{ é linear}}{=} G(\lambda F(u))$

$\overset{G \text{ é linear}}{=} \lambda G(F(u))$

$= \lambda (G \circ F)(u)$, $\forall \lambda \in \mathbb{R}, \forall u \in \mathcal{U}$

$\therefore G \circ F$ é linear

Ex.: $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$(x,y) \longmapsto F(x,y) = (x+y, x-y, 2x+3y)$

$G: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$(x,y,z) \longmapsto G(x,y,z) = (x+z, y-z)$ $\quad \exists\, G \circ F$? e $F \circ G$?

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & \mathbb{R}^3 \arrow[r, "G"] & \mathbb{R}^2 \end{tikzcd}$$

a) $(G \circ F)(x,y) \overset{def}{=} G(F(x,y)) = G(x+y,\ x-y,\ 2x+3y)$

$= ((x+y)+(2x+3y),\ (x-y)-(2x+3y))$

$= (3x+4y,\ -x-4y)$

b)

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "G"] \arrow[rr, bend right, "G \circ F"'] & \mathbb{R}^2 \arrow[r, "F"] & \mathbb{R}^3 \end{tikzcd}$$

$(F \circ G)(x,y,z) = F(G(x,y,z)) = F(x+z,\ y-z)$

$= ((x+z)+(y-z),\ (x+z)-(y-z),\ 2(x+z)+3(y-z))$

$= (x+y,\ x-y+2z,\ 2x+3y-z)$

2-) Sejam $F, G \in L(M_2(\mathbb{R}))$

$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-d & b+c \\ b+c & a-d \end{pmatrix}$ e

$G\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-d & b-c \\ b-c & a-d \end{pmatrix}$

a) Determine $F \circ G$ e $G \circ F$

b) Prove que $Im(F) = Ker(G)$

a) $(F \circ G)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{def}{=} F\left(G\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = F\begin{pmatrix} a-d & b-c \\ b-c & a-d \end{pmatrix} = \begin{pmatrix} 0 & 2b-2c \\ 2b-2c & 0 \end{pmatrix}$

$(G \circ F)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = G\left(F\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = G\begin{pmatrix} a-d & b+c \\ b+c & a-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

É o conjunto de vetores que cai no vetor nuleo

$F: U \to V$

$Ker(F) = \{u \in U \mid F(u) = 0_V\}$

$\therefore Ker(G \circ F) = M_2(\mathbb{R})$

b)

1-) $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in Im(F)$ se existe $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

tal que $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-d & b+c \\ b+c & a-d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \Rightarrow \begin{cases} e = h \\ f = g \end{cases}$

$A \in Im(F)$ se $A = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = e\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + f\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Para determinar se as matrizes é LI, $\Rightarrow$ combinação Linear igual ao vetor nulo.

$\therefore Im(F) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$

2-) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Ker(G)$ se existe

$G\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-d & b-c \\ b-c & a-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} a = d \\ b = c \end{cases}$

$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad Ker(G) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = Im(F)$

Transformação Linear

Matriz

Sejam U e V espaços vetoriais sobre $\mathbb{R}$
Consideremos as bases:

$B = \{u_1, u_2, \dots, u_n\}$ de U e

$C = \{v_1, v_2, \dots, v_m\}$ de V

Seja $F: U \to V$ transformação linear dada. A matriz de F em relação às bases dadas é obtida do seguinte modo:

Calculamos F em cada um dos vetores da base B. Os vetores obtidos $F(u_1), F(u_2), \dots, F(u_n)$ estão em V. Como $v_1, v_2, \dots, v_m$ é base de V, cada $F(u_j)_{j=1,2,\dots,n}$ que está em V é combinação linear dos vetores da base C de V, isto é:

$$F(u_1) = a_{11}v_1 + a_{21}v_2 + \dots + a_{m1}v_m$$
$$F(u_2) = a_{12}v_1 + a_{22}v_2 + \dots + a_{m2}v_m$$
$$\vdots \qquad\qquad \vdots$$
$$F(u_n) = a_{1n}v_1 + a_{2n}v_2 + \dots \; a_{mn}v_m$$

Cada CL é um sistema linear
Resolvidos todos os sistemas teremos os a_{ij} que determinam a matriz do tipo $m \times n$

$$\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$$

Que é denominada matriz de F em relação as bases B e C. Indicamos: $(F)^B_C$ ou $(F)_{B,C}$

Exemplo:

Dada a TL $F: \mathbb{R}^3 \to \mathbb{R}^2 \mid F(x,y,z) = (x-3y+z,\ 2x+y-2z)$
Determinar a matriz de F em relação às bases:

1º) Canônicas

2º) $B = \{(1,0,1),\ (0,-1,1),\ (0,0,-1)\}\ \mathbb{R}^3$
$C = \{(1,1),\ (1,2)\}\ \mathbb{R}^2$

1º) Bases canônicas

$B = \{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}$
$C = \{e_1 = (1,0),\ e_2 = (0,1)\}$

$F(e_1) = F(1,0,0) = (1,2) = a_{11}(1,0) + a_{21}(0,1)$
$F(e_2) = F(0,1,0) = (-3,1) = a_{12}(1,0) + a_{22}(0,1)$
$F(e_3) = F(0,0,1) = (1,-2) = a_{13}(1,0) + a_{23}(0,1)$

$\begin{cases} a_{11} = 1 \\ a_{21} = 2 \end{cases} \quad \begin{cases} a_{12} = -3 \\ a_{22} = 1 \end{cases} \quad \begin{cases} a_{13} = 1 \\ a_{23} = -2 \end{cases}$

R: $(F)_{B,C} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}_{2\times 3}$

2°)

$F(1,0,1) = (2,0) = a_{11}(1,1) + a_{21}(1,2)$

$F(0,-1,1) = (4,-3) = a_{12}(1,1) + a_{22}(1,2)$

$F(0,0,-1) = (-1,2) = a_{13}(1,1) + a_{23}(1,2)$

$\begin{cases} a_{11} + a_{21} = 2 \\ a_{11} + 2a_{21} = 0 \end{cases} \sim \begin{cases} a_{11} + a_{21} = 2 \\ -a_{21} = 2 \end{cases}$ $\quad a_{11} = 2+2 = 4$, $a_{21} = -2$

$\begin{cases} a_{12} + a_{22} = 4 \\ a_{12} + 2a_{22} = -3 \end{cases} \sim \begin{cases} a_{12} + a_{22} = 4 \\ -a_{22} = 7 \end{cases}$ $\quad a_{12} = 4+7 = 11$, $a_{22} = -7$

$\begin{cases} a_{13} + a_{23} = -1 \\ a_{13} + 2a_{23} = 2 \end{cases} \sim \begin{cases} a_{13} + a_{23} = -1 \\ -a_{23} = -3 \end{cases}$ $\quad a_{13} = -1-3 = -4$, $a_{23} = 3$

$$R: (F)_{B,C} = \begin{bmatrix} 4 & 11 & -4 \\ -2 & -7 & 3 \end{bmatrix}_{2\times3}$$

Exercício

Dadas as bases

$B = \{(1,1,1), (0,1,1), (1,0,1)\}$ $\mathbb{R}^3$ $C = \{(1,-1), (1,2)\}$ $\mathbb{R}^2$ e a matriz de $F: \mathbb{R}^3 \to \mathbb{R}^2$ (linear) em relação às bases dadas, Determinar $F(x,y,z)$

$$(F)_{B,C} = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \end{pmatrix}$$

$F(x,y,z) = ?$ $\quad (x,y,z) \in \mathbb{R}^3$

$1(1,-1)-(1,2) = (0,-3)$

$0(1,-1)+2(1,2) = (2,4)$

$3(1,-1)+(1,2) = (4,-1)$

$F(1,1,1) = (0,-3)$

$F(0,1,1) = (2,4)$

$F(1,0,1) = (4,-1)$

$(x,y,z) = a(1,1,1)+b(0,1,1)+c(1,0,1)$

$$\begin{cases} a + \quad + c = x \\ a+b \qquad = y \\ a+b+c = z \end{cases} \sim \begin{bmatrix} 1 & 0 & 1 & x \\ 1 & 1 & 0 & y \\ 1 & 1 & 1 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -1 & y-x \\ 0 & -1 & 0 & x-z \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -1 & y-x \\ 0 & 0 & -1 & y-z \end{bmatrix} \sim \begin{cases} a + \quad + c = x \\ \quad b - c = y-x \\ \quad -c = y-z \end{cases} \quad \begin{matrix} a = x+y-z \\ b = y-x+c = -x+z \\ c = z-y \end{matrix}$$

$F(x,y,z) = aF(1,1,1)+bF(0,1,1)+cF(1,0,1)$

$= a(0,-3)+b(2,4)+c(4,-1)$

$= (x+y-z)(0,-3)+(-x+z)(2,4)+(z-y)(4,-1)$

$= (0,-3x-3y+3z)+(-2x+2z,-4x+4z)+(4z-4y,-z+y)$

$= (-2x-4y+6z, -7x-2y+6z)$

$F: \mathbb{R}^3 \to \mathbb{R}^2 \mid F(x,y,z) = (-2x-4y+6z, -7x-2y+6z)$

Produto Interno

$V = \mathbb{R}^3$

$u = (u_1, u_2, u_3)$

$v = (v_1, v_2, v_3)$

$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$

$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{u \cdot u}$

Seja V um e.v. real

Def.: Uma operação:

$V \times V \longrightarrow \mathbb{R}$

$(u,v) \longmapsto \langle u,v \rangle$ satisfazendo:

i) $\langle u,v \rangle = \langle v,u \rangle \quad \forall u,v \in V$

ii) $\langle \alpha u, v \rangle = \alpha \langle u,v \rangle, \ \forall \alpha \in \mathbb{R}, \ \forall u,v \in V$

iii) $\langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle, \ \forall u_1, u_2 \text{ e } v \in V$

iv) $\langle u,u \rangle \geq 0, \ \forall u \in V$

$\langle u,u \rangle = 0 \iff u = 0$ é chamado "produto interno"

Obs.: Se existe um p.i. definido sobre V ele é dito um "Espaço Euclidiano"

Ex 1: $V = \mathbb{R}^n$; $x = (x_1, x_2, \dots x_n)$

$y = (y_1, y_2, \dots, y_n)$

$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$(x,y) \longmapsto \langle x,y \rangle = x_1y_1 + \dots + x_ny_n$ é um produto interno sobre $\mathbb{R}^n$.

Exercício Prove!

Ex 2 $V = \mathbb{R}^3$

$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x, y) \longmapsto \langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$ define um p. i. sobre $\mathbb{R}^3$?

i) $\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3 = y_1 x_1 + y_2 x_2 - y_3 x_3 = \langle y, x \rangle$

ii) $\langle \alpha x, y \rangle = (\alpha x_1) y_1 + (\alpha x_2) y_2 - (\alpha x_3, y_3)$
$= \alpha (x_1 y_1 + x_2 y_2 - x_3 y_3) = \alpha \langle x, y \rangle$

iii) $\langle u + v, y \rangle = (u_1 + v_1) y_1 + (u_2 + v_2) y_2 - (u_3 + v_3) y_3$
$= (u_1 y_1 + u_2 y_2 - u_3 y_3) + (v_1 y_1 + v_2 y_2 - v_3 y_3) = \langle u, y \rangle + \langle v, y \rangle$

$u = (u_1, u_2, u_3)$
$v = (v_1, v_2, v_3)$
$y = (y_1, y_2, y_3)$

$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

iv) $x = (x_1, x_2, x_3)$
$\langle x, x \rangle = x_1^2 + x_2^2 - x_3^2$

Por exemplo

$x = (0, 0, 1) \neq \vec{0}$ $\therefore$ Não vale a iv

$\langle x, x \rangle = -1 < 0$

Ex 3: $V = M_{m \times n}(\mathbb{R})$

Obs: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} \end{bmatrix}$

Traço de uma matriz é:

Traço de A: $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

/ /

Definimos:

$$M_{m\times n}(\mathbb{R}) + M_{m\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(A=(a_{ij}),\ B=(b_{ij})) \longmapsto \langle A,B\rangle = tr(B^tA)$$

<u>Exercício</u>: Mostre que ela define um p.i. sobre $M_{m\times n}(\mathbb{R})$.

<u>Ex</u>: $M_{2\times 3}(\mathbb{R})$

$$A=\begin{bmatrix}1 & 0 & -1\\ 2 & 1 & 3\end{bmatrix} \qquad B=\begin{bmatrix}4 & -1 & 0\\ 2 & 3 & 4\end{bmatrix}$$

$$B^tA = \underbrace{\begin{bmatrix}4 & 2\\ -1 & 3\\ 0 & 4\end{bmatrix}}_{3\times 2}\underbrace{\begin{bmatrix}1 & 0 & -1\\ 2 & 1 & 3\end{bmatrix}}_{2\times 3} = \begin{bmatrix}8 & * & *\\ * & 3 & *\\ * & * & 12\end{bmatrix}$$

a) $\langle A,B\rangle = tr(B^tA) = 8+3+12 = 23$

$A_{m\times n};\ B_{m\times n}$

$(B^tA)_{n\times n}$

b) $\langle A,A\rangle$ $$\begin{bmatrix}1 & 2\\ 0 & 1\\ -1 & 3\end{bmatrix} \times \begin{bmatrix}1 & 0 & -1\\ 2 & 1 & 3\end{bmatrix} = \begin{bmatrix}5 & * & *\\ * & 1 & *\\ * & * & 10\end{bmatrix}$$

$\langle A,A\rangle = tr(A^tA) = 5+1+10 = 16$

c) $\langle B, A\rangle = tr(A^t B)$

$$A^t B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & -1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 8 & * & * \\ * & 3 & * \\ * & * & 12 \end{bmatrix}$$

$\langle B, A\rangle = tr(A^t B) = 23$

Seja $V = P^n(t)$; definimos $\rightarrow B = (1, t, t^2, t^3 + \dots + t^n)$

$P^n(t) \times P^n(t) \longrightarrow \mathbb{R}$

$(p, q) \longmapsto \langle p, q\rangle = \int_0^1 p(t)\, q(t)\, dt$

Obs: $p \in P^n(t) \Rightarrow p(t) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Ex: Se $n = 2$: $P^2(t)$

a) $p(t) = 1 - t = 0.t^2 + (-1)t + 1.1$

$q(t) = t + t^2 = 1.t^2 + 1.t + 0.1$

$\langle p(t), q(t)\rangle = \int_0^1 (1-t)(t+t^2)\, dt$

$\int_0^1 t + t^2 - t^2 - t^3\, dt = \int_0^1 (t - t^3)\, dt = \frac{t^2}{2} - \frac{t^4}{4} \Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

b) Se $p(t) = 2t + 3$ calcule $\langle p(t), q(t)\rangle$

$q(t) = t - t^2$

$\langle p(t), q(t)\rangle = \int_0^1 (2t+3)(t-t^2)\, dt = \int_0^1 (-2t^3 - t^2 + 3t)\, dt = -\frac{t^4}{2} - \frac{t^3}{3} + \frac{3t^2}{2} \Big|_0^1$

$\langle p(t), q(t)\rangle = -\frac{1}{2} - \frac{1}{3} + \frac{3}{2} = \frac{-3-2+9}{6} = \frac{4}{6} = \frac{2}{3}$

A operação define um p.i sobre $P^n(t)$:

i) $<p(t), q(t)> = \int_0^1 p(t)\, q(t)\, dt = \int_0^1 q(t)\, p(t)\, dt = <q(t)\, p(t)>$

ii) $<\alpha p(t), q(t)> = \int_0^1 \alpha p(t)\, q(t)\, dt$
$= \alpha \int_0^1 p(t)\, q(t)\, dt = \alpha <p(t), q(t)>$
$\forall \alpha \in \mathbb{R},\ \forall p, q \in P^n(t)$

iii) $<p_1(t) + p_2(t), q(t)> = \int_0^1 (p_1(t) + p_2(t))\, q(t)\, dt$
$= \int_0^1 (p_1(t)\, q(t)) + (p_2(t)\, q(t))\, dt$
$= \int_0^1 p_1(t)\, q(t)\, dt + \int_0^1 p_2(t)\, q(t)\, dt$
$= <p_1(t), q(t)> + <p_2(t), q(t)>$

iv) $<p(t), p(t)> = \int_0^1 p(t) \cdot p(t)\, dt$
$= \int_0^1 p^2(t)\, dt \geq 0,\ \forall p(t) \in P^n(t)$
$<p(t), p(t)> = 0 \iff p(t) = 0$

Transformação linear matriz e operações

172-) $F: \mathbb{R}^3 \to \mathbb{R}^2$ linear

Matriz de F em relação às bases

$B = \{(1,1,1), (1,1,0), (1,0,0)\}$ $\mathbb{R}^3$ $F-$

$C = \{(1,1), (1,0)\}$ $\mathbb{R}^2$

$$(F) = \begin{pmatrix} 3 & 5 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

Determinar:

a) $F(x,y,z) = ?$

b) Base $\ker(F) = ?$

c) Matriz F na base canônica

177-) Dadas as matrizes de $F, G: \mathbb{R}^3 \to \mathbb{R}^3$

$$(F) = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & -2 \end{bmatrix}, \quad (G) = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & -3 \\ 2 & -3 & 0 \end{bmatrix}$$ na base canônica

Determinar matriz $G \circ F$ na base canônica e a função $(G \circ F)(x,y,z) = ?$

170-) Determinar uma base do núcleo e da imagem de $F: \mathbb{R}^3 \to M_2(\mathbb{R})$

Dada por $F(x,y,z) = \begin{pmatrix} x-y & y-z \\ z-y & x-z \end{pmatrix}$

151-) Sendo F e G lineares $F: \mathbb{R}^2 \to \mathbb{R}^2$ e $G: \mathbb{R}^2 \to \mathbb{R}$ tais que $F(2,1) = (1,2)$ e $G(3,6) = 7$ Calcular $(G \circ F)(4,2)$

192) $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ linear definido por $T(f) = xf' - f''$ Determinar a matriz de T na base $A = \{1, x, x^2, x^3\}$

177-) Propriedade

$[G \circ F] = [G].[F]$

Não esquecer

Foi determinada a matriz na base canônica. É apenas observar a matriz

$F(x,y,z) = (x + 2y - z, 2x + 3z, -x + 3y - 2z)$

$$[G \circ F] = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & -3 \\ 2 & -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -4 & 6 & -7 \\ 4 & -11 & 10 \\ -4 & 4 & -11 \end{bmatrix}$$

172-) $3(1,1) + (1,0) = (4,3)$

$5(1,1) + 2(1,0) = (7,5)$

$1(1,1) + 0(1,0) = (1,1)$

$F(1,1,1) = (4,3)$

$F(1,1,0) = (7,5)$

$F(1,0,0) = (1,1)$

$(x,y,z) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$$\begin{cases} x = a+b+c \\ y = a+b \\ z = a \end{cases} \qquad \begin{array}{l} c = x-a-b = x-\cancel{z}-y+\cancel{z} = x-y \\ b = y-z \end{array}$$

$\therefore\ a = z \quad b = y-z \quad c = x-y$

$F(x,y,z) = aF(1,1,1) + bF(1,1,0) + cF(1,0,0)$

$F(x,y,z) = z(4,3) + (y-z)(7,5) + (x-y)(1,1)$

$= (4z+7y-7z+x-y\ ,\ 3z+5y-5z+x-y)$

$F(x,y,z) = (x+6y-3z,\ x+4y-2z)$

b) Base do Nucleo

$F(u) = 0$

$$\begin{cases} x+6y-3z = 0 \\ x+4y-2z = 0 \end{cases} \rightarrow \begin{cases} x+6y-3z = 0 \\ 2y-z = 0 \end{cases}$$

$z = 2y \qquad x = 3z-6y$

$= 2.2y-6y = 4y-6y = -2y$

$(x,y,z) = (-2y,\ y,\ 2y)$

$= y(-2,1,2)$

$\therefore$ Base do Nucleo: $\{(-2,1,2)\}$

151-) $(G o F)(4,2) = ?$

$$(GoF)(4,2) = G(F(4,2))$$
$$= G(F(2(2,1)))$$
$$= 2G(F(2,1))$$
$$= 2G(1,2)$$
$$= 2 \cdot \frac{7}{3} = \frac{14}{3}$$

$G(3,6): G(3(1,2)) = 7$
$= 3G(1,2) = 7$
$= G(1,2) = \frac{7}{3}$

$(GoF)(\lambda u) = \lambda (GoF)(u); \lambda \in \mathbb{R}$

170-)

$$F(x,y,z) = \begin{pmatrix} x-y & y-z \\ z-y & x-z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} x-y=0 \\ y-z=0 \\ z-y=0 \\ x-z=0 \end{cases} \sim \begin{cases} x-y=0 \\ y-z=0 \\ 0=0 \\ -y+z=0 \end{cases} \sim \begin{cases} x-y=0 \\ y-z=0 \end{cases} \quad \begin{matrix} x=y \\ y=z \end{matrix}$$

Base do Nucleo $(F) = \{(1,1,1)\}$

$$F(1,0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$F(0,1,0) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$F(0,0,1) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\underbrace{\dim V}_{3} = \underbrace{\dim Nuc(F)}_{1} + \underbrace{\dim ImF(F)}_{2}$$

$$(GoF)(\lambda u) = \alpha \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{cases} -\alpha = 1 \Rightarrow \alpha = -1 \\ \alpha - \beta = 0 \\ -\alpha + \beta = 0 \\ -\beta = +1 \Rightarrow \beta = -1 \end{cases}$$

Base $Im(F) = \left\{ \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\}$

192.)

$T(f) = xf' - f''$

$A = \{1, x, x^2, x^3\}$

$f=1;\ f=x;\ f=x^2;\ f=x^3$

$T(1) = x.0 - 0 = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$

$T(x) = x.1 - 0 = x = 0.1 + 1.x + 0.x^2 + 0.x^3$

$T(x^2) = x.2x - 2 = 2x^2 - 2x$

$= (-2).1 + 0.x + 2.x^2 + 0.x^3$

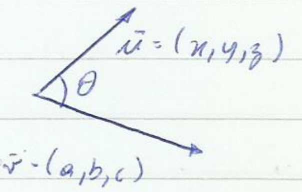
$T(x^3) = x.3x^2 - 6x = 3x^3 - 6x$

$= 0.1 + 0.1 - 6.x + 3.x^3$

$$T = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Produto Interno

Na Geometria Analítica temos que



$\vec{u} \cdot \vec{v} = ax + by + cz$

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$

Seja V um espaço vetorial sobre $\mathbb{R}$

Sejam $u, v \in V$

O produto interno de u e v, denotado por $\langle u, v \rangle$, denotado por $\langle u, v \rangle$, é uma aplicação $V \times V$ em $\mathbb{R}$

$\langle u, v \rangle \quad V \times V \to \mathbb{R}$

$\quad (u, v) \to \langle u, v \rangle$

tal que:

1) $\langle u, v \rangle = \langle v, u \rangle$

2) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$

$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

3) $\langle \lambda u, v \rangle = \langle u, \lambda v \rangle = \lambda \langle u, v \rangle$

4) $\langle u, u \rangle \geq 0$ e $\langle u, u \rangle = 0 \iff u = 0 \quad \forall u, v, w \in V$

Nota:

Em um espaço vetorial V pode se definir diferentes produtos internos

Produtos Internos Usuais

I. Seja $V = \mathbb{R}^n$

$u = (x_1, x_2, \dots, x_n)$

$v = (y_1, y_2, \dots, y_n)$

$\langle u, v \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

II. Seja $V = P_n(\mathbb{R})$

$p(x) \in V$

$q(x) \in V$

$\langle p(x), q(x) \rangle = \int_0^1 p(x) q(x)\, dx$

$\langle p(x), q(x) \rangle = \int_0^1 p(x) q(x)\, dx$ é um produto interno

De fato

1) $\langle p(x), q(x) \rangle = \int_0^1 p(x) q(x)\, dx$
$= \int_0^1 q(x) p(x)\, dx = \langle q(x), p(x) \rangle$

2) $p(x), q(x), r(x) \in P_n(x)$

$$\begin{aligned} \langle p(x), q(x) + r(x) \rangle &= \int_0^1 p(x)[q(x) + r(x)] \\ &= \int_0^1 p(x) q(x) + p(x) r(x)\, dx \\ &= \int_0^1 p(x) q(x)\, dx + \int_0^1 p(x) r(x)\, dx \\ &= \langle p(x), q(x) \rangle + \langle p(x), r(x) \rangle \end{aligned}$$

3-) $p(x)$ e $q(x) \in \mathbb{P}_n(x)$ e $\lambda \in \mathbb{R}$

$$\langle \lambda p(x), q(x) \rangle = \int_0^1 \lambda p(x) q(x)\,dx$$
$$= \lambda \int_0^1 p(x) q(x)\,dx$$
$$= \lambda \langle p(x), q(x) \rangle$$

4-) $p(x) \in \mathbb{P}_n(\mathbb{R})$

$$\langle p(x), p(x) \rangle = \int_0^1 p(x)p(x)\,dx = \int_0^1 [p(x)]^2\,dx \geq 0$$

III - Seja $V = M_{m\times n}(\mathbb{R})$

$A, B \in V$

$\langle A, B \rangle = \underbrace{tr(B^t A)}_{\text{Soma dos elementos da diagonal principal}}$ → traço B^tA

Se $B, A \in M_{n\times m}(\mathbb{R})$ então $\langle A, B \rangle = tr(B^t A)$

Exemplos

1) Sejam $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ e $B = \begin{pmatrix} -1 & 2 & 3 \\ 3 & 1 & -2 \end{pmatrix}$

Calcule $\langle A, B \rangle$

$$B^tA = \begin{bmatrix} -1 & 3 \\ 2 & 1 \\ 3 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 7 \\ 5 & 4 & 0 \\ 4 & 6 & -7 \end{bmatrix}$$

$\langle A, B \rangle = tr(B^tA) = 1 + 4 - 7 = \underline{-2}$

2) $V = P_2(\mathbb{R})$

Dados $p(t) = t^2+t$ e $q(t) = t-1$ determine $\langle p(t), q(t) \rangle$

$$\langle p(t), q(t) \rangle = \int_0^1 (t^2+t)(t-1)\,dt$$
$$= \int_0^1 t^3 \cancel{-t^2} \cancel{+t^2} - t\,dt$$
$$= \frac{1}{4}t^4 - \frac{1}{2}t^2 \Big|_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

3) Seja $V = \mathbb{R}^4$

Determine $\langle u, v \rangle$ onde $u = (1,1,1,-1)$

$v = (1,2,1,1)$

$$\langle u, v \rangle = 1.1 + 1.2 + 1.1 + 1.(-1) = 3$$

Espaço Euclidiano

Um espaço vetorial V, munido de um produto interno é dinominado espaço vetorial euclidiano

Norma de um vetor

Seja V um espaço vetorial euclidiano e $u \in V$

A norma de um vetor u, denotada por $\|u\|$, é definida por

$\|u\| = \sqrt{\langle u, v \rangle}$

Exemplo: $V = M_{2\times3}(\mathbb{R})$

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$ determine $\|A\|$

$||A|| = \sqrt{<A,A>}$

$<A,A> = tr(A^t A)$

$$A^t A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 0 \\ 4 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

$<A,A> = 5+4+5 = 14$

$||A|| = \sqrt{<A,A>} = \sqrt{14}$

2) Seja $V = P_2(\mathbb{R})$ e $q(t) = t-1$ Determine $||q(t)||$.

$||q(t)|| = \sqrt{<q(t), q(t)>}$

$$<q(t), q(t)> = \int_0^1 q(t)q(t)\,dt = \int_0^1 (t-1)^2\,dt$$
$$= \int_0^1 t^2 - 2t + 1\,dt = \frac{1}{3}t^3 - t^2 + t \Big|_0^1 =$$
$$= \frac{1}{3} - 1 + 1 = \frac{1}{3}$$

$$||q(t)|| = \sqrt{<q(t); q(t)>} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

3) Seja $V = \mathbb{R}^4$ e $u = (1,1,1,-1)$ Determine $||u||$

$$||u|| = \sqrt{<u,u>} = \sqrt{<(1,1,1,-1),(1,1,1,-1)>}$$
$$= \sqrt{1+1+1+1} = \sqrt{4} = 2$$

Propriedades Seja $u, v, w \in V$ (espaço euclidiano)

1) $\|u\| \geq 0$ e $\|u\| = 0 \iff u = 0$

2) $\|\lambda v\| = |\lambda| \|v\|$, $\lambda \in \mathbb{R}$

3) $\|u + v\| \leq \|u\| + \|v\|$

4) $|\langle u, v \rangle| \leq \|u\| \|v\|$

Nota:

Seja V um espaço euclidiano de dimensão finita
Dois vetores $u, v \in V$ são ortogonais se $\langle u, v \rangle = 0$
[Se $\langle u, v \rangle = 0 \Rightarrow u$ e v são ortogonais]

Distância entre vetores de um espaço euclidiano V

Sejam $u, v \in V$
A distância entre u e v, denotada por $d(u,v)$, é definido por $d(u,v) = \|u - v\|$

Exemplos:

1) $V = M_{2\times3}(\mathbb{R})$

Sejam $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ e $B = \begin{pmatrix} -1 & 2 & 3 \\ 3 & 1 & -2 \end{pmatrix}$ determine $d(A,B)$

$$d(A,B) = \| A-B \| = \sqrt{\langle A-B, A-B \rangle} = \sqrt{\langle C, C \rangle} = \sqrt{tr(C^t C)}$$

$$C = A - B = \begin{pmatrix} 3 & 0 & -4 \\ -2 & -1 & 4 \end{pmatrix}$$

$$C^t C = \begin{bmatrix} 3 & -2 \\ 0 & -1 \\ -4 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & -4 \\ -2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 2 & -20 \\ 2 & 1 & -4 \\ -20 & -4 & 32 \end{bmatrix}$$

$$d(A,B) = \sqrt{13+1+32} = \sqrt{46}$$

2) $V = P_n(\mathbb{R})$

Sejam $p(t) = t^2+t$, $q(t) = t-1$ determine $d(p(t), q(t))$

$$d(p(t),q(t)) = \| p(t)-q(t) \| = \sqrt{\langle p(t)-q(t), p(t)-q(t) \rangle}$$

onde $\langle p(t)-q(t), p(t)-q(t) \rangle = \langle t^2+1, t^2+1 \rangle$

$$\langle t^2-1, t^2-1 \rangle = \int_0^1 (t^2+1)^2 dt = \int_0^1 (t^4+2t^2+1)\, dt =$$
$$= \frac{1}{5}t^5 + \frac{2}{3}t^3 + t \Big|_0^1 = \frac{1}{5} + \frac{2}{3} + 1 = \frac{28}{15}$$

$$d(p(t), q(t)) = \sqrt{\frac{28}{15}}$$

Propriedades Seja V um espaço euclidiano

1) $d(u,v) \geq 0 \quad \forall u,v \in V$
2) $d(u,v) = d(v,u)$
3) $d(u,v) \leq d(u,w) + d(w,v)$

Complemento Ortogonal

Seja V um espaço vetorial munido de um produto interno e seja W um subespaço vetorial de V

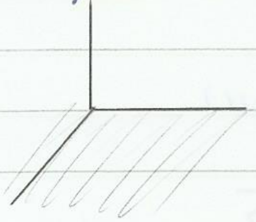
O complemento ortogonal de W, denotado por $W^\perp$, é definido por

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0, \ \forall w \in W \}$$

__Exemplo:__

Seja $V = \mathbb{R}^3$ e $W = \{ (0,0,z) \text{ tal que } z \in \mathbb{R} \}$

$W^\perp = \{ (x,y,0) \mid x, y \in \mathbb{R} \}$



Auto valor e Auto vetor

ou valor e vetor característico

ou valor e vetor próprio

V: Espaço vetorial sobre um corpo $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ ou $\mathbb{K}=\mathbb{C}$)

$T: V \to V$ um operador linear

Definição: Um escalar $\lambda \in \mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ ou $\mathbb{K}=\mathbb{C}$) é denominado valor próprio de T se existe $v \in V$, $v \neq 0$, tal que $T(v) = \lambda v$

Seja a matriz quadrada de ordem n associada ao operador obtida em relação à base canônica

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

OBS: A definição também é dada da seguinte forma: $\boxed{Av = \lambda v}$

1ª parte

Determinar os valores próprios

a) Matriz característica $A - \lambda I_n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn}-\lambda \end{bmatrix} = A - \lambda I_n$$

b) Polinômio característico: $p(\lambda)$

$p(\lambda) = \det(A - \lambda I_n)$
Genericamente

$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + 0$

Se $\det(A - \lambda I_n)$ for ordem $= 2 \Rightarrow p(\lambda)$ grau $= 2$
" " " " $= 3$ " " $= 3$
e assim por diante

c) Equação característica $p(\lambda) = 0$

Se $p(\lambda)$ tem grau $= 2$ Então Eq. 2º grau
Se $p(\lambda)$ " " $= 3$ " " 3º grau

Os valores próprios são as raízes da equação característica

2º parte

Determinar os vetores próprios para cada valor próprio está associado um vetor próprio $v_i \neq 0$ $(i = 1, 2, \dots, n)$ obtido ao resolver o sistema linear homogêneo

$$\begin{pmatrix} a_{11} - \lambda_i & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda_i \cdots & a_{2n} \\ a_{n1} & a_{n2} & a_{nn} - \lambda_i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

onde A Matriz coluna $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

credeal

É a matriz das coordenadas do vetor $v_i = (x_1, x_2, \dots, x_n) \neq 0$ que é a solução do sistema.

Exemplo: Dado o operador linear $T: \mathbb{R}^2 \to \mathbb{R}^2$ tal que $T(x,y) = (-3x - 2y, 5x + 4y)$

Determinar

a) Valores próprios

b) Vetores próprios

a)

$B = \{(1,0), (0,1)\}$

$T(1,0) = (-3, 5)$

$T(0,1) = (-2, 4)$

$$A = \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$$

Matriz característica $A - \lambda I_2$

$$\begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -3-\lambda & 5 \\ -2 & 4-\lambda \end{bmatrix}$$

Polinômio característico $P(\lambda)$

$$P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} -3-\lambda & 5 \\ -2 & 4-\lambda \end{vmatrix} = (-3-\lambda)(4-\lambda) - (5 \cdot (-2))$$

$$P(\lambda) = \lambda^2 - \lambda - 12 + 10$$

$$= \lambda^2 - \lambda - 2$$

$$\lambda^2 - \lambda - 2 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} \qquad \lambda_1 = 2 \quad \lambda_2 = -1$$

Valores próprios $\lambda_1 = -1$ e $\lambda_2 = 2$

/ /

b) Vetores próprios

vetor próprio $v_1 = (x, y) \neq 0$ para $\lambda_1 = -1$

$$\begin{bmatrix} -3-(-1) & -2 \\ 5 & 4-(-1) \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -2x - 2y = 0 \\ 5x + 5y = 0 \end{cases} \therefore x = -y$$

Para (Admitindo) $x = 1 \therefore y = -1$

$\therefore v_1 = (1, -1)$

Vetor próprio $v_2 = (x, y) \neq 0$ para $\lambda_2 = 2$

$$\begin{bmatrix} -3-(2) & -2 \\ 5 & 4-(2) \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -2 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -5x - 2y = 0 \\ 5x + 2y = 0 \end{cases} \therefore \quad -5x - 2y = 0$$

$$-5x = 2y \quad \therefore y = \frac{-5x}{2}$$

Para $x = 2 \therefore y = -5$

$\therefore v_2 = (2, -5)$

$(x, -\frac{5}{2}x)$

$x(1, -\frac{5}{2})$

Aplicação de valores próprios

Teorema de Cayley - Hamilton

Toda matriz é um zero de seu polinômio característico

$$P(A) = 0$$

$P(\lambda) = \lambda^2 - \lambda - 2$

$P(A) = A^2 - A - 2I$

$$P(A) = \begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix}^2 - \begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Aplicação do teorema

1) Calcular A^4

2) " A^{-1}

1-) Calcular A^4

$P(A) = 0$

$A^2 - A - 2I = 0 \Rightarrow A^2 = A + 2I$

$A^3 = A^2 A = (A+2I)A = A^2 + 2A = A + 2I + 2A = 3A + 2I$

$A^4 = A^3 A = (3A+2I)A = 3A^2 + 2A = 3(A+2I) + 2A = 5A + 6I$

$$= 5\begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & -10 \\ 25 & 20 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} -9 & -10 \\ 25 & 26 \end{bmatrix}$$

Calculo A^{-1}

$P(A) = 0$

$A^2 - A - 2I = 0$

$A^{-1}(A^2 - A - 2I) = A^{-1}0$

$A - A^{-1}A - 2A^{-1}I = 0$

$A - I - 2A^{-1} = 0$

$2A^{-1} = A - 1 \quad \therefore \quad A^{-1} = \frac{1}{2}(A-1)$

$$A - I = \begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 5 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & -2 \\ 5 & 3 \end{pmatrix}$$

Exercícios

1.) Seja $V = C[-1,1]$ o ev das funções reais contínuas, munido do p.i $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$. Dadas as funções $f(x) = 3x + 2$ e $g(x) = 5x^2 + x + 2$, determine α para que $f(x) \cdot g(x)$ sejam ortogonais. Resp. $\alpha = \frac{-13}{6}$

2.) Dadas as TL

$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto F(x,y) = (2x - 3y,\ y + x)$ e $G \circ F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (G \circ F)(x,y) = (2x - 3y,\ -2y + 3x)$, determinar

a) $G(x,y)$
b) $G^{-1}(x,y)$

3.) Seja $V = C[0,1]$ (ev das funções reais contínuas definidas no intervalo $[0,1]$. Para f e g em V, definimos o p.i $\langle f(t), g(t) \rangle = \int_0^1 f(t)\, g(t)\, dt$. Dadas: $f(t) = e^t$ e $g(t) = -e^{-t}$, calcule

a) $\langle f, g \rangle$; b) $\| f \|$ c) $\| g \|$ d) $\| f + g \|$

4.) Dada a matriz

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix}, \text{ determinar}$$

a) polinômio característico
b) valores próprios
c) vetores próprios

5. Sabendo-se que T é uma TL do R^3 tal que

$T(\overbrace{1,1,0}^{e_1}) = (1,1,1)$

$T(\underbrace{0,-1,1}_{e_2}) = (0,-1,-1)$ determinar:

$T(\underbrace{1,0,2}_{e_3}) = (1,-1,0)$

i) $T(x,y,z)$

ii) $T^{-1}(x,y,z)$

iii) a matriz de T^{-1} em relação à base canônica

3-) $\langle f,g \rangle = \langle e^t, -e^{-t} \rangle = \int_0^1 e^t \cdot (-e^{-t})\, dt = -\int_0^1 dt = -t \Big|_0^1 = -1$

b) $\|f\| = \sqrt{\langle f,f \rangle}$

$\langle f,f \rangle = \langle e^t, e^t \rangle = \int_0^1 e^{2t}\, dt = \int_0 e^u \frac{1}{2} du = \frac{1}{2} e^{2t} \Big|_0^1 = \frac{e^2}{2} - \frac{1}{2}$

$u = 2t$

$du = 2dt$

$\|f\| = \sqrt{\frac{e^2-1}{2}}$

c) $\|g\| = \sqrt{\langle g,g \rangle}$

$\langle g,g \rangle = \langle -e^{-t}, -e^{-t} \rangle = \int_0^1 e^{-2t}\, dt = \int_0^1 -e^s \frac{1}{2} d = -\frac{1}{2} e^{-2t} \Big|_0^1$

$s = -2t$ $\quad = -\frac{1}{2} e^{-2} + \frac{1}{2} = -\frac{1}{2e^2} + \frac{1}{2} = \frac{-1+e^2}{2e^2}$

$ds = -2dt$

$\|g\| = \sqrt{\frac{e^2-1}{2e^2}} = e^{-1}\sqrt{\frac{e^2-1}{2}}$

d) $\|f+g\| = \sqrt{\langle f+g, f+g \rangle}$

$\langle f+g, f+g \rangle = \int_0^1 (e^t - e^{-t})^2 \, dt = \int_0^1 e^{2t} - 2 + e^{-2t} \, dt$

$= \frac{1}{2} e^{2t} - 2t - \frac{1}{2} e^{-2t} \Big|_0^1 = \frac{1}{2} e^2 - 2 - \frac{1}{2} e^{-2} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}$

$= \frac{1}{2} e^2 - 2 - \frac{1}{2} e^{-2}$

$\|f+g\| = \sqrt{\frac{1}{2} e^2 - 2 - \frac{1}{2} e^{-2}}$

4-)

Valor próprio e vetor próprio

Exercícios

1) Dado o operador linear $T: \mathbb{R}^3 \to \mathbb{R}^3 \mid T(x,y,z) = (x, -2x-y, 2x+y+2z)$

Determinar:

a) Valores próprios

b) Vetores próprios

2) Determinar o operador linear $T: \mathbb{R}^3 \to \mathbb{R}^3$ Dados os valores e os respectivos vetores próprios.

$\lambda_1 = 1 \quad v_1 = (1,1,0)$

$\lambda_2 = -1 \quad v_2 = (0,1,1)$

$\lambda_3 = 2 \quad v_3 = (0,0,1)$

(Definição do vetor próprio $T(v) = \lambda v \mid v \neq 0$)

Respostas

1-) (Primeiro, calcular com a base canônica)

$T(1,0,0) = (1,-2,2)$

$T(0,1,0) = (0,-1,1)$

$T(0,0,1) = (0,0,2)$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A - \lambda I_3 = \begin{bmatrix} 1-\lambda & 0 & 0 \\ -2 & -1-\lambda & 0 \\ 2 & 1 & 2-\lambda \end{bmatrix}$$

$P(\lambda) = \det(A - \lambda I_3)$

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & -1-\lambda & 0 \\ 2 & 1 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-1-\lambda)(2-\lambda)$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2$$

$P(\lambda) = 0$

$(1-\lambda)(-1-\lambda)(2-\lambda) = 0$ $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \\ \lambda_3 = 2 \end{cases}$ valores próprios

Valores próprios p/ $\lambda = 1$

$$\begin{bmatrix} 1-1 & 0 & 0 \\ -2 & -1-1 & 0 \\ 2 & 1 & 2-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -2x - 2y = 0 \\ 2x + y + z = 0 \end{cases} \sim \begin{cases} -2x - 2y = 0 \\ -y + z = 0 \end{cases} \therefore \begin{matrix} x = -y \\ y = z \end{matrix}$$

p/ $y = 1$ $x = -1$ e $z = 1$

$v_1 = (-1, 1, 1)$

Vetor próprio p/ $\lambda = -1$

$$\begin{bmatrix} 1-(-1) & 0 & 0 \\ -2 & -1-(-1) & 0 \\ 2 & 1 & 2-(-1) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2x = 0 \\ -2x = 0 \\ 2x + y + 3z = 0 \end{cases} \quad \therefore \quad x = 0 \qquad y + 3z = 0 \therefore \quad y = -3z$$

Para $z = 1$ $\quad y = -3$

$v_2 = (0, -3, 1)$

Vetor próprio p/ $\lambda = 2$

$$\begin{bmatrix} 1-2 & 0 & 0 \\ -2 & -1-2 & 0 \\ 2 & 1 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ -2 & -3 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{cases} -x = 0 \\ -2x - 3y = 0 \\ 2x + y = 0 \end{cases} \quad \begin{matrix} x = 0 \\ y = 0 \end{matrix}$$

$v_3 = (0, 0, z) \,/\, z \neq 0 \quad$ p/ $z = 1 \therefore v_3 = (0, 0, 1)$

/ /

2) Aplicando a definição

$\boxed{T(v) = \lambda v}$

$T(v_1) = \lambda_1 v_1 \Rightarrow T(v_1) = (1,1,0) \qquad T(1,1,0) = (1,1,0)$

$T(v_2) = \lambda_2 v_2 \Rightarrow T(v_2) = (0,-1,-1) \qquad T(0,1,1) = (0,-1,-1)$

$T(v_3) = \lambda_3 v_3 \Rightarrow T(v_3) = (0,0,2) \qquad T(0,0,1) = (0,0,2)$

$T(v) = ?$

$\forall v = (x,y,z) \in \mathbb{R}^3$ é C.L. de v_1, v_2, v_3 é uma base

$(x,y,z) = \alpha(1,1,0) + \beta(0,1,1) + \gamma(0,0,1)$

$$\begin{cases} \alpha = x \\ \alpha + \beta = y \\ \beta + \gamma = z \end{cases} \sim \begin{cases} \alpha = x \\ -\beta = x - y \\ \gamma = x - y + z \end{cases} \quad \beta = y - x$$

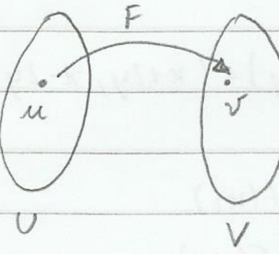
$T(x,y,z) = ?$

$$\begin{aligned} T(x,y,z) &= T(\alpha(1,1,0) + \beta(0,1,1) + \gamma(0,0,1)) \\ &= \alpha T(1,1,0) + \beta T(0,1,1) + \gamma T(0,0,1) \\ &= x(1,1,0) + (y-x)(0,-1,-1) + (x-y+z)(0,0,2) \\ &= (x,x,0) + (0, x-y, x-y) + (0,0,2x-2y+2z) \\ &= (x, 2x-y, 3x-3y+2z) \end{aligned}$$

Transformação Linear

→ $F: U \to V$

Isto indica que F é uma função ou aplicação de U em V, que associa cada elemento de $u \in U$, a um único elemento de $v \in V$



$F: U \to V$

→ contra domínio

→ domínio

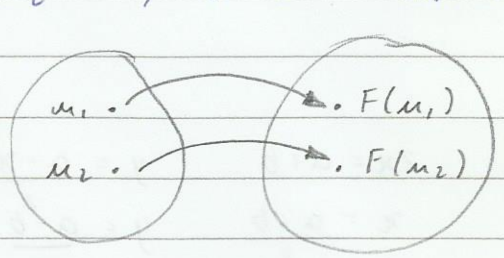
→ função ou aplicação

→ A Imagem de F (ImF) é o subconjunto do contradomínio (V)

$ImF = \{v \in V \mid \exists u \in U \text{ e } v = F(u)$

→ $F: U \to V$ é igual a $G: U \to V$ se
$F(u) = G(u)$; $\forall u \in U$

→ Uma aplicação (Função) é injetora se:
$\forall u_1, u_2 \in U$, $u_1 \neq u_2 \Rightarrow F(u_1) \neq F(u_2)$



→ Uma aplicação $F: U \to V$ é sobrejetora, se e somente se,
$\forall v \in V$, $\exists u \in U \mid v = F(u)$
ou seja a imagem de F, seja igual ao contradomínio (V)

/ /

→ Uma aplicação é bijetora se e somente se, a aplicação for injetora e sobrejetora. Define-se a aplicação da inversa:

$F^{-1}: V \to U$ tal que $F^{-1}(F(u)) = u$ e $F(F^{-1}(v)) = v, \forall u \in U$

Exemplos:

a) $F: \mathbb{R}^2 \to \mathbb{R}^2$, definida por $F(u) = F(x,y) = (x+2y, x-2y)$

Injetora: $\forall u_1, u_2 \in U, u_1 \neq u_2 \Rightarrow F(u_1) \neq F(u_2)$

e $u_1 = u_2 \Rightarrow F(u_1) = F(u_2)$

$F(u_1) = F(x_1, y_1) = (x_1 + 2y_1, x_1 - 2y_1)$

$F(u_2) = F(x_2, y_2) = (x_2 + 2y_2, x_2 - 2y_2)$

Como $F(u_1) = Fu_2 \Rightarrow F(x_1, y_1) = F(x_2, y_2)$ então

$x_1 + 2y_1 = x_2 + 2y_2$

$x_1 - 2y_1 = x_1 - 2y_1$ $\therefore$ É injetora

b) $F: \mathbb{R}^2 \to \mathbb{R}^2$, definida por $F(u) = F(x,y) = (x+2y, x-2y)$

Sobrejetora: $\forall v = (a,b) \in V$

$\forall v = (a,b) \in V$ $\begin{cases} x + 2y = a \\ x - 2y = b \end{cases}$ $2x = a+b$ $x = \dfrac{a+b}{2}$ $y = \dfrac{a-x}{2}$ $y = \dfrac{a-b}{4}$

$\therefore \forall v \in V, u \in U \mid v = F(u)$.

c) $F: \mathbb{R}^2 \to \mathbb{R}^2$, definida por $F(u) = F(x,y) = (x+2y, x-2y)$

$F^{-1}: V \to U \quad F^{-1}(F(u)) = u \quad e \quad F(F^{-1}(v)) = v$

$F^{-1}(x,y) = (a,b) \Rightarrow F^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{4}\right)$

* $F: U \to V$ é uma transformação linear de U em V se:

a) $\forall u_1, u_2 \in U \Rightarrow F(u_1 + u_2) = F(u_1) + F(u_2)$
b) $\forall u_1 \in U; \forall \lambda \in \mathbb{R} \Rightarrow F(\lambda u_1) = \lambda F(u_1)$

1-) $F: \mathbb{R} \to \mathbb{R}$, definida por $F(x) = Ax$; $A \in \mathbb{R}$

a) $\forall u_1 \in \mathbb{R} \mid u_1 = x_1 \Rightarrow F(u_1) = Ax_1$
$\forall u_2 \in \mathbb{R} \mid u_2 = x_2 \Rightarrow F(u_2) = Ax_2$

$F(u_1 + u_2) = Ax_1 + Ax_2 = A(x_1 + x_2) = F(u_1) + F(u_2)$

b) $\forall u_1 = x_1 \in \mathbb{R}; \forall \lambda \in \mathbb{R}$

$F(\lambda u_1) = F(\lambda x_1) = \lambda A x_1 = \lambda(Ax_1) = \lambda F(u_1)$ $\therefore$ É Linear

2-) $F: \mathbb{R} \to \mathbb{R}$, definida por $F(x) = Ax + B$; $A, B \in \mathbb{R}$

a) $\forall u_1 \in \mathbb{R} \mid u_1 = x_1 \Rightarrow F(u_1) = Ax_1 + B$
$\forall u_2 \in \mathbb{R} \mid u_2 = x_2 \Rightarrow F(u_2) = Ax_2 + B$
$F(u_1 + u_2) = Ax_1 + B + Ax_2 + B$
$= A(x_1 + x_2) + B \neq F(u_1) + F(u_2)$ $\therefore$ Não é Linear

/ /

3.) $F: \mathbb{R}^2 \to \mathbb{R}^2$ definido por $F(x,y) = (x+y, x-y)$

a) $\forall u_1 = (x_1, y_1) \in \mathbb{R}^2 \Rightarrow F(u_1) = (x_1 + y_1, x_1 - y_1)$
$\forall u_2 = (x_2, y_2) \in \mathbb{R}^2 \Rightarrow F(u_2) = (x_2 + y_2, x_2 - y_2)$

$F(u_1 + v_2) = (x_1 + y_1 + x_2 + y_2, x_1 - y_1 + x_2 - y_2) = F(u_1) + F(u_2)$

b) $\forall u_1 = (x_1, y_1) \in \mathbb{R}; \forall \lambda \in \mathbb{R} \Rightarrow$

$$F(\lambda u_1) = F(\lambda x_1, \lambda y_2) = (\lambda x_1 + \lambda y_1, \lambda x_1 - \lambda y_1)$$
$$= (\lambda(x_1 + y_1), \lambda(x_1 - y_1))$$
$$= \lambda(x_1 + y_1, x_1 - y_1)$$
$$F(\lambda u_1) = \lambda F(u_1)$$

$\therefore$ Transformação Linear

4.) $F: \mathbb{R}^2 \to \mathbb{R}^2$; definida por $F(x,y) = (x+y, x-y+1)$

a) $\forall u = (x_1, y_1) \in \mathbb{R}^2 \Rightarrow F(u) = (x_1 + y_1, x_1 - y_1 + 1)$
$\forall v = (x_2, y_2) \in \mathbb{R}^2 \Rightarrow F(v) = (x_2 + y_2, x_2 - y_2 + 1)$

$F(u+v) = (x_1 + y_1 + x_2 + y_2, x_1 + x_2 - y_1 - y_2 + 1) \neq F(u) + F(v)$

$\therefore$ Não é transformação linear

* Se $U = V$, a transformação linear $F: U \to V$, denomina-se operador linear em U

Proposições:

1-) $F(0)=0$

$F(0) = F(0+0) = F(0)+F(0)$

$\therefore \quad F(0) = F(0)+F(0) \qquad \{-F(0)\}$

$F(0)-F(0) = F(0)+F(0)-F(0)$

logo $\quad 0 = F(0)$

2-) $F(-u) = -F(u) \ ; \ \forall u \in U$

$F(-u) = F(-1u) = -1F(u) = -F(u)$

3-) $F(u_1-u_2) = F(u_1)-F(u_2) \ ; \ \forall u_1, u_2 \in U$

$F(u_1-u_2) = F(u_1+(-u_2)) = F(u_1)-F(u_2)$

4-) Se W é subespaço vetorial de U, então $F(W)$ é subespaço vetorial de V

a) $F(0)=0$, logo $0 \in F(W)$

b) $\begin{cases} \forall w_1 \in W \Rightarrow F(w_1) \in F(W) \\ \forall w_2 \in W \Rightarrow F(w_2) \in F(W) \end{cases}$

$F(w_1+w_2) = \underbrace{F(w_1)}_{\in F(W)} + \underbrace{F(w_2)}_{\in FW} \quad \in F(W)$

c) $\forall w_1 \in W, \forall \lambda \in \mathbb{R} \Rightarrow F(w_1) \in F(W) \Rightarrow F(\lambda w_1) = \lambda \underbrace{F(w_1)}_{\in FW} \in W$

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Teorema: Exemplo

$F: \mathbb{R}^3 \to \mathbb{R}^3$, sabendo:

$F(1,0,0) = (1,1,0)$

$F(0,2,0) = (2,0,6)$

$F(0,1,-1) = (2,-2,3)$

$\forall u = (x,y,z) \in \mathbb{R}^3$ ∴ os vetores contido no $\mathbb{R}^3$ forma uma combinação linear

$(x,y,z) = a(1,0,0) + b(0,2,0) + c(0,1,-1)$; $a,b,c \in \mathbb{R}$

$$\begin{cases} x = a & \therefore\ a = x \\ y = 2b + c & \therefore\ b = (y-c)/2 = (+y+z)/2 \\ z = -c & \therefore\ c = -z \end{cases}$$

$F(x,y,z) = F[a(1,0,0)] + F[b(0,2,0)] + F[c(0,1,-1)] =$

$= a\,F(1,0,0) + b\,F(0,2,0) + c\,F(0,1,-1)$

$= x(1,1,0) + (y+z)\frac{(2,0,6)}{2} + z(2,-2,3)$

$F(x,y,z) = \underline{(x+y-2z,\ x+2z,\ 3y)}$

Ex: 2 $\quad F: \mathbb{R}^2 \to \mathbb{R}^3$, sabendo

$F(1,1) = (1,0,-1)$

$F(0,1) = (2,3,4)$

$\forall u = (x,y) \in \mathbb{R}^2 \qquad (x,y) = a(1,1) + b(0,1)$

$$\begin{cases} x = a & a = x \\ y = a + b & b = y - x \end{cases}$$

$F(x,y) = a\,F(1,1) + b\,F(0,1)$
$= x\,(1,0,-1) + (y-x)\,(2,3,4)$
$= (x-2x+2y\,,\ -3x+3y\,,\ -x-4x+4y)$
$= \underline{(-x+2y\,,\ -3x+3y\,,\ -5x+4y)}$

Conferindo

$F(1,1) = (-1+2\,,\ -3+3\,,\ -5+4) = (1,0,-1)$
$F(0,1) = (0+2\,,\ 0+3\,,\ 0+4) = (2,3,4)$

$\therefore\ F(x,y) = (-x+2y\,,\ -3x+3y\,,\ -5x+4y)$

Exercícios Propostos

128-) Determinar $F(x,y)$, sabendo:
F é um operador linear do $\mathbb{R}^2$
$F(1,1) = (1,3)$ e $F(1,2) = (-1,1)$

Como F é um operador linear do $\mathbb{R}^2$
$F: \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = a(1,1) + b(1,2)$

$$\begin{cases} x = a+b & a = x-b \Rightarrow a = x-(y-x) = 2x-y \\ y = a+2b & y = x-b+2b \therefore b = y-x \end{cases}$$

$F(x,y) = a\,F(1,1) + b\,F(1,2)$
$= (2x-y)\,(1,3) + (y-x)\,(-1,1)$
$= (2x-y-y+x\,,\ 6x-3y+y-x)$
$= (3x-2y\,,\ 5x-2y)$

(/ /)

129-) Determine $T: \mathbb{R}^3 \to \mathbb{R}^3$ linear de tal forma que: $T(1,0,1) = (-1,-1,0)$
$T(0,1,-1) = (-1,1,2)$ e $T(0,0,-1) = (0,-1,3)$

$(x,y,z) = a(1,0,1) + b(0,1,-1) + c(0,0,-1)$

$$\begin{cases} x = a - c & \Rightarrow a = x \\ y = b + c & \Rightarrow b = y \\ z = a - b - c & \Rightarrow c = a - b - z = x - y - z \end{cases}$$

$F(x,y,z) = aF(1,0,1) + bF(0,1,-1) + cF(0,0,-1)$
$= x(-1,-1,0) + y(-1,1,2) + (x-y-z)(0,-1,3)$
$= (-x-y, -x+y-x+y+z, 2y+3x-3y-3z)$

$F(x,y,z) = (-x-y, -2x+2y+z, 3x-y-3z)$

130-) Seja $T: P_2(\mathbb{R}) \to P_3(\mathbb{R}) \mid T(p(x)) = p(x) + x^2 \cdot p'(x)$, onde $p'(x)$ é a derivada de $p(x)$. Mostre que T é uma transformação linear

$T(p(x)) = p(x) + x^2 \cdot p'(x)$

a) $\forall u \in \mathbb{R} \mid u = p_1(x_1) \Rightarrow F(u) = p_1(x_1) + x_1^2 \, p_1'(x)$
$\forall v \in \mathbb{R} \mid v = p_2(x_2) \Rightarrow F(v) = p_2(x_2) = x_2^2 \, p_2'(x_2)$

$p(u+v) = p_1(x_1) + p_2(x_2) + (p_1(x_1) + p_2(x_2))^2 \, (p_1(x_1)\, p_2(x_2))'$
$= p_1(x_1) + p_2(x_2) + p_1^2(x_1) + 2p_1(x_1)p_2(x_2) + p_2^2(x_1) + p_1'(x_1)p_2(x_2) + p_1(x_1)p_2'(x_1)$
$=$

131-) Seja $V = C[0,1]$ o espaço vetorial das funções reais contínuas definidas em $[0,1]$ e $W = \mathbb{R}$. Mostre que $L: V \to W \mid L(f) = \int_0^1 f(x)\,dx$ é uma T.L.

132-) Determine $F: \mathbb{R}^3 \to \mathbb{R}^2$ linear sabendo que $F(1,0,1) = (1,-1)$; $F(0,1,1) = (2,3)$ e $F(0,0,1) = (0,2)$

$$(x,y,z) = a(1,0,1) + b(0,1,1) + c(0,0,1)$$

$$\begin{cases} x = a \\ y = b \\ z = a + b + c \end{cases} \qquad \begin{aligned} a &= x \\ b &= y \\ c &= -x - y + z \end{aligned}$$

$$\begin{aligned} F(x,y,z) &= a\,F(1,0,1) + b\,F(0,1,1) + c\,F(0,0,1) \\ &= x(1,-1) + y(2,3) + (-x-y+z)(0,2) \end{aligned}$$

$$\begin{aligned} \therefore F(x,y,z) &= (x + 2y,\ -x + 3y - 2x - 2y + 2z) \\ &= (x + 2y,\ -3x + y + 2z) \end{aligned}$$

/ /

133) Determine $F(x,y)$, operador linear do $\mathbb{R}^2$ tal que $F(1,-1) = (-1,2)$ e $F(1,0) = (1,3)$

$F: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) = a(1,-1) + b(1,0)$

$\begin{cases} x = a + b \\ y = -a \end{cases}$ $\quad b = x + y$ $\quad a = -y$

$F(x,y) = aF(1,-1) + bF(1,0)$
$= -y(-1,2) + (x+y)(1,3)$
$= (y + x + y, 2y + 3x + 3y)$

$F(x,y) = (x + 2y, 3x + 5y)$

134-) Seja F um operador linear do $\mathbb{R}^2$ tal que $F(1,0) = (2,1)$ e $F(0,1) = (1,4)$

a) $F(2,4) = ?$

$F: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) = a(1,0) + b(0,1)$ $\quad F(x,y) = aF(1,0) + bF(0,1)$

$\begin{cases} x = a \\ y = b \end{cases}$ $\quad = x(2,1) + y(1,4)$

$F(x,y) = (2x + y, x + 4y)$

$\therefore F(2,4) = (2.2 + 4, 2 + 4.4)$
$= (8, 18)$

b) o vetor $(x,y) \in \mathbb{R}^2$ tal que $F(x,y) = (2,3)$

$F(x,y) = (2x + y, x + 4y)$ $\quad \begin{cases} 2x + y = 2 \\ x + 4y = 3 \end{cases} \sim \begin{cases} 2x + y = 2 \\ -7y = -4 \end{cases}$

$F(x,y) = (2,3)$

$y = \frac{4}{7}$ $\quad x = \frac{10}{14} = \frac{5}{7}$ $\quad (x,y) = \left(\frac{5}{7}, \frac{4}{7}\right)$

135-) Verifique se F é linear, nos seguintes casos:

A) $F: M_{m \times n} \to M_{n \times m}$; $F(A) = A^t$

a) $\forall u =$

B) $F: \mathbb{R}^2 \to \mathbb{R}$; $F(x,y) = xe^{2t} + ye^t$; $t \in \mathbb{R}$

a) $\forall u = (x_1, y_1) \Rightarrow F(u) = x_1 e^{2t} + y_1 e^t$

$\forall v = (x_2, y_2) \Rightarrow F(v) = x_2 e^{2t} + y_2 e^t$

$$F(u+v) = (x_1 + x_2) e^{2t} + (y_1 + y_2) e^t$$
$$= \underbrace{x_1 e^{2t} + y_1 e^t}_{F(u)} + \underbrace{x_2 e^{2t} + y_2 e^t}_{F(v)}$$

$\therefore F(u+v) = F(u) + F(v)$

b) $\forall u = (x_1, y_1) \in \mathbb{R}$, $\forall \lambda \in \mathbb{R}$

$$F(\lambda u) = F(\lambda(x_1, y_1)) = \lambda x_1 e^{2t} + \lambda y_1 e^t$$
$$= \lambda \underbrace{(x_1 e^{2t} + y_1 e^t)}_{F(u)}$$

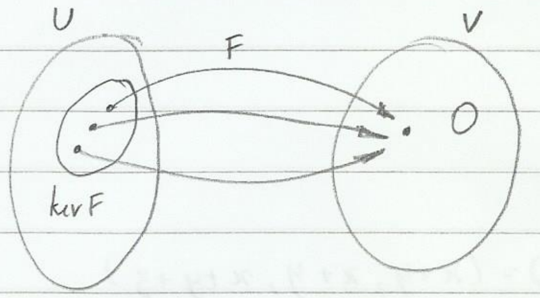
$\therefore F(\lambda u) = \lambda F(u)$

c) $F: M_2(\mathbb{R}) \to \mathbb{R}$; $F = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+b+c+d$

Núcleo de uma transformação linear

O núcleo (ou kernel) da transformação linear é o conjunto a seguir

$\ker F = \{ u \in U \mid F(u) = 0 \}$



Exemplos:

1-) $F: \mathbb{R}^2 \to \mathbb{R}^2$, definida por $F(x,y) = (x+y, x+y)$

$\ker F = \{ u \in U \mid F(u) = 0 \}$

$F(0,0) = (x+y, x+y)$

$\begin{cases} x+y = 0 & x = -y \\ x+y = 0 & x - \end{cases}$

$\ker F = \{(x,y)\}$
$= \{(-y,y)\}$
$= \{y(-1,1)\}$

→ $F: \mathbb{R}^3 \to \mathbb{R}^3$, $F(x,y,z) = (x+y, x-z, x+y+z)$

$\ker F = \{ u \in U \mid F(u) = 0 \}$

$F(0,0,0) = (x+y,\ x-z,\ x+y+z)$

$$\begin{cases} x+y=0 & x=-y \\ x-z=0 & z=x=-y \\ x+y+z=0 & z=0 \end{cases}$$

$\ker F = (-y, 2y, -y)$

$= y(-1,2,-1)$

$\ker F = \{(-1,2,-1)\}$

➤ $F: \mathbb{R}^3 \to \mathbb{R}^3$, $F(x,y,z) = (x+y,\ x+y,\ x+y+z)$

$\ker F = \{u \in U \mid F(u) = 0\}$

$\therefore F(0,0,0) = (x+y,\ x+y,\ x+y+z)$

$$\begin{cases} x+y=0 & x=-y \\ x+y+z & z=-x-y=-(-y)-y=y-y=0 \end{cases}$$

$\ker F = (-y, y, 0)$

$= y(-1,1,0)$

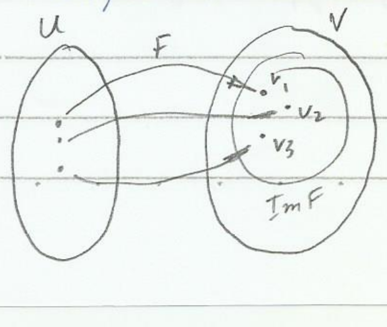
$= \{(-1,1,0)\}$

Obs.: O kernel (o núcleo) de uma transformação linear $F: U \to V$ é um subespaço vetorial de U.

Imagem de uma transformação linear

Dada uma transformação linear $F: U \to V$, o conjunto imagem é:

$\operatorname{Im} F = \{v \in V \mid v = F(u);\ u \in U\} = \{F(u) \in V \mid u \in U\}$



OBSERVAÇÕES:

$F: U \to V$

+ A imagem de F é uma transformação linear sobrejetora
+ A transformação linear é injetora se $\ker F = \{0\}$
+ $\dim U = \dim(\ker F) + \dim(\operatorname{Im} F)$
+ Se U e V são esp. vet. sobre $\mathbb{R}$ com mesma dimensão: F é sobrejetora $\iff$ F é bijetora $\iff$ F é injetora e também F transforma toda base vetorial U em uma base do espaço vetorial V.

Exemplo: $F: \mathbb{R}^3 \to \mathbb{R}^3$, definida por $F(x,y,z) = (x, x+y, x+y+z)$

$\to$ Determine uma base, dimensão $\ker F$ e $\operatorname{Im} F$

$\ker F = \{u \in U \mid F(u) = 0\}$

$(0,0,0) = (x, x+y, x+y+z)$

$$\begin{cases} x = 0 & \to x = 0 \\ x+y = 0 & \to y = 0 \\ x+y+z = 0 & \to z = 0 \end{cases}$$

$\therefore \ker F = \emptyset$

$\dim(\ker F) = 0$

$\operatorname{Im} F = \{v \in V \mid v = F(u);\ u \in U\}$

$\dim U = \dim(\ker F) + \dim(\operatorname{Im} F)$

$\dim U = 0 + 3 = 3$

$v = (x, x+y, x+y+z)$

$\quad = x(1,1,1) + y(0,1,1) + z(0,0,1)$

$\operatorname{Im} F = [(1,1,1), (0,1,1), (0,0,1)]$

$\operatorname{Im} F = \{(1,1,1), (0,1,1), (0,0,1)\}$

$\therefore$ a $\dim(\operatorname{Im} F) = 3$

/ /

136-) $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, definida por:

$$T(ax^2+bx+c) = (a+2b)x + (b+c)$$

a-) $u = -4x^2+2x-2 \in ker(T) = ?$

~~$Ker(t) = \{p \in P_2 \mid T(p)=0,\ p \in \mathbb{R}\}$~~

(No momento estou sem borracha)

~~$0x^2+0x+0 = (a+2b)x + (b+c)$~~

~~$\begin{cases} a+2b=0 & a=-2b=2c \\ b+c=0 & b=-c \end{cases}$~~

$-4x^2+2x-2 = (-4+2.2)x + (-2+2) = 0$

Como $Ker(t) = \{p \in P_2 \mid T(p)=0\}$ $\therefore$ o vetor $-4x^2+2x-2$ pertence a $Ker(t)$.

b) $v = x^2+2x+1 \in Im(T) = ?$

$Im\,t = \{p_1 \in P_2 \mid p_2 = T(p_1)\}$

$x^2+2x+1 = (1+2.2)x + (2+1)$

$\neq 5x+3$ $\therefore$ O vetor $v = x^2+2x+1$ não pertence a $Im(T)$

c) $0x^2+0x+0 = (a+2b)x + (b+c)$

$\begin{cases} a+2b=0 & a=2c \\ b+c=0 & b=-c \end{cases}$ Admitindo $c=1 \to a=2$ e $b=-1$

$\therefore$ o vetor $2x^2-x+1 \in Ker(t)$

a $\dim(Ker(t)) = 1$

d) $Im(t) = \{ v \in V \mid \exists u \in U \text{ e } v = F(u) \}$

ou melhor

$Im\ T = \{ F(u) \in V \mid u \in U \}$

$T(ax^2 + bx + c) = (a + 2b)x + (b + c)$

$T(1,0,0) = x$

$T(0,1,0) = 2x + 1$

$T(0,0,1) = 1$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Base da $Im(t) = \{(1,0),(0,1)\}$

$dim(Im(F)) = \underline{2}$

E) Injetora é quando

$\forall u, v \in U,\ u \neq v \Rightarrow F(u) \neq F(v)$

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Exercícios propostos

137.) $F: \mathbb{R}^4 \to \mathbb{R}^3$ $F(x,y,z,w) = (x-y+w, 2x+y-z+w, y+z-w)$,

→ determine uma base

→ " dimensão da imagem e do núcleo de F

$\ker F = \{u \in U \mid F(u) = 0\}$

$F(x,y,z,w) = 0$

$F(0,0,0,0) = (x-y+w, 2x+y-z+w, y+z-w)$

$$\begin{cases} x-y+w=0 \\ 2x+y-z+w=0 \\ y+z-w=0 \end{cases} \to \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & -4 & 2 \end{bmatrix}$$

$$\begin{cases} x-y+w=0 \\ 3y-z-w=0 \\ -4z+2w=0 \end{cases} \qquad w = \frac{4z}{2} = 2z \qquad y = \frac{w+z}{3} = \frac{2z+z}{3} = z \qquad x = y-w \qquad x = z-2z = -z$$

$(x,y,z,w) = (-z, z, z, 2z) = z(-1,1,1,2)$

Base do Núcleo $\{(-1,1,1,2)\}$ e dim = 1

$\dim U = \dim(\ker F) + \dim(\operatorname{Im} F)$

$\quad 4 \qquad\quad 1 \qquad\qquad 3$

Admitindo: $F(1,0,0,0) = (1,2,0)$

$F(0,1,0,0) = (-1,1,1)$

$F(0,0,1,0) = (0,-1,1)$

$F(0,0,0,1) = (1,1,-1)$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Base da Imagem: $\{(1,2,0),(0,3,1),(0,0,4)\}$ dim Im $F = 3$

138-) $F: \mathbb{R}^4 \to \mathbb{R}^2$, dada por $F(x,y,z,t) = (x+2y, z-t)$

$\ker F = \{u \in U \mid F(u) = 0\}$

$F(0,0,0,0) = (x+2y, z-t)$

$\begin{cases} x+2y=0 & x=-2y \\ z-t=0 & z=t \end{cases}$

$\ker F = (-2y, y, z, z)$

$= y(-2,1,0,0) + z(0,0,1,1)$

$= [(-2,1,0,0),(0,0,1,1)]$

$= \{(-2,1,0,0),(0,0,1,1)\}$ ∴ dim $= 2$

$F(1,0,0,0) = (1,0)$

$F(0,1,0,0) = (2,0)$

$F(0,0,1,0) = (0,1)$

$F(0,0,0,1) = (0,-1)$

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Base Im$F = \{(1,0),(0,1)\}$ ∴ dim $= 2$

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139.) $F: \mathbb{R}^3 \to \mathbb{R}^4 \mid F(x,y,z) = (x+2z,\ x+2y,\ y-z,\ 2x+4z)$ e
$G: \mathbb{R}^4 \to \mathbb{R}^4 \mid G(x,y,z,t) = (x-y+t,\ t-y,\ z+t,\ 2x-2y+2t)$

- Determine: uma base da $Im(F)$, $Im(G)$, $S = Im(F) + Im(G)$

$F(1,0,0) = (1,1,0,2)$
$F(0,1,0) = (0,2,1,0)$
$F(0,0,1) = (2,0,-1,4)$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Base da $Im(F) = \{(1,1,0,2), (0,2,1,0)\}$

$G(1,0,0,0) = (1,0,0,2)$
$G(0,1,0,0) = (-1,-1,0,-2)$
$G(0,0,1,0) = (0,0,1,0)$
$G(0,0,0,1) = (1,1,1,2)$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Base da $Im(G) = \{(1,0,0,2), (0,-1,0,0), (0,0,1,0)\}$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & -1 & 4 \\ 1 & 0 & 0 & 2 \\ -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Base de $S = \{(1,1,0,2), (0,2,1,0), (0,0,1,0)\}$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Base de $S: \{(1,0,0,2), (0,-1,0,0), (0,0,1,0)\}$

140-) $B = [(1,2,3,4); (3,0,1,1); (2,-2,-2,-3)]$ $\quad T: \mathbb{R}^4 \to \mathbb{R}^3$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 1 \\ 2 & -2 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -8 & -11 \\ 0 & -6 & -8 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -8 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$B = \{(1,2,3,4), (0,-6,-8,-11)\}$ $\quad \dim = 2$

141-) $T: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad T(x,y,z) = (x-y+2z, 2x+y, -x-2y+2z)$

$u = (a,b,c) \in \operatorname{Im} T$

$(a,b,c) = (x-y+2z, 2x+y, -x-2y+2z)$

$$\begin{cases} x-y+2z = a \\ 2x+y = b \\ -x-2y+2z = c \end{cases} \sim \begin{bmatrix} 1 & -1 & 2 & a \\ 2 & 1 & 0 & b \\ -1 & -2 & 2 & c \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 3 & -4 & b-2a \\ 0 & -3 & 4 & a+c \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 3 & -4 & b-2a \\ 0 & 0 & 0 & -a+b+c \end{bmatrix} \therefore \quad -a+b+c = 0$$

$$a = b+c$$

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142-) $\mathbb{R}^2$: $F(x,y) = (0, 2y-x)$
- determinar a dimensão do núcleo

$Ker(F) = \{u \in U \mid F(u) = 0\}$

$F(0,0) = (0, 2y-x)$
$2y - x = 0 \therefore x = 2y$

Base de $Ker(F) = \{(2y, y)\}$
$= \{y(2,1)\}$
$= \{(2,1)\} \therefore \dim ker(F) = 1$

143-) $L: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ $\quad L(at^3 + bt^2 + ct + d) = (a-b)t^3 + (c-d)t$

A) determinar base de $Ker(L) = ?$ e da $Im(L)$
B) O vetor $u = t^3 + t^2 + t - 1 \in Ker(L) = ?$
C) O vetor $v = 3t^2 - t \in Im(L) = ?$

A) $Ker: \{u \in U \mid F(u) = 0\}$

$L(0t^3 + 0t^2 + 0t + 0) = (a-b)t^3 + (c-d)t$

$\begin{cases} a - b = 0 & a = b \\ c - d = 0 & c = d \end{cases}$ Gerador: $[(a, a, c, c)]$
$[a(1,1,0,0) + c(0,0,1,1)]$
Base: $\{(1,1,0,0), (0,0,1,1)\}$

$L(1,0,0,0) = t^3$
$L(0,1,0,0) = -t^3$
$L(0,0,1,0) = t$
$L(0,0,0,1) = -t$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \therefore B = \{(1,0,0,0), (0,0,1,0)\}$

B) $L(t^3+t^2+t-1) = (1-1)t^3 + (1-(-1))t$

$= 2t$ $\therefore$ Não pertence ao Ker

C) $L(3t^2-1) = (0-3)t^3 + (0-(-1))t$

$L(3t^2-1) = -3t^3 + t$

R: Pertence, pois $Im(f) = \{v \in V \mid \exists u \in U \text{ e } v = F(u)\}$

Transformação Linear

Para ser transformação linear, deve respeitar a seguinte relação

a) $\forall u_1, u_2 \in U \Rightarrow F(u_1+u_2) = F(u_1) + F(u_2)$

b) $\forall u_1 \in U$; $\forall \lambda \in R \Rightarrow F(\lambda u) = \lambda(F(u_1))$

$F: U \to V$ — contradomínio: V; domínio: U

Aplicação ou "Lei"

$\dim U = \dim Nuc(F) + \dim Im(F)$

Imagem é o subconjunto do contradomínio

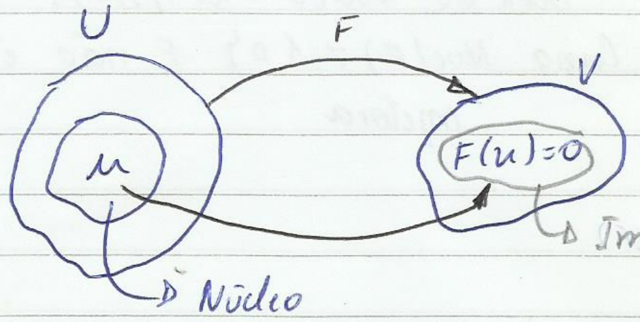
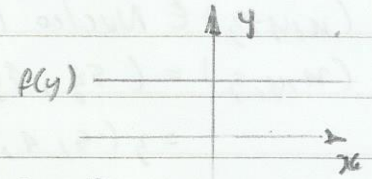


Imagem $ImF \{v \in V \,/\, \exists u \in U \text{ e } v = F(u)\}$



D = R Imagem f(y)

Isomorfismo : - O núcleo é (0) $Nucle(F) = \{0\}$

Injetora

$\forall u_1, u_2 \in U$, $u_1 \neq u_2 \Rightarrow F(u_1) \neq F(u_2)$

$u_1 = u_2 \Rightarrow F(u_1) = F(u_2)$

Sobrejetora: $\forall v \in V$, $\exists u \in U \,/\, v = F(u)$

se $F: U \to V$ for injetora e sobrejetora é isomorfismo

se $F: U \to U$ for injetora e sobrejetora é automorfismo

/ /

1) Dada TL $F: \mathbb{R}^3 \to \mathbb{R}^2 \mid F(x,y,z) = (x-y-z, 2x-3y+2z)$

operador

a) base do núcleo e dimensão

$F(u)=0$

$F(x,y,z) = (0,0)$

Para determinar a base do núcleo, usamos a seguinte definição $F(u)=0$

$(0,0) = (x-y-z, 2x-3y+2z)$

$$\begin{cases} x-y-z=0 \\ 2x-3y+2z=0 \end{cases} \sim \begin{cases} x-y-z=0 \\ 0-y+4z=0 \end{cases} \quad \begin{aligned} x &= z+4z = 5z \\ y &= 4z \end{aligned}$$

$(x,y,z) \in$ Nucleo /

$(x,y,z) = (5z, 4z, z)$

$= z(5,4,1)$

$\therefore$ Base do Nucleo $= \{(5,4,1)\}$

Como $Nuc(F) \neq \{0\}$ F não é injetora

b) Base da Imagem e a dim (F)

$Im\,F = \{u \in U \mid \exists v \in V \text{ e } v = F(u)\}$

Para determinar a base da Imagem, utiliza a base canônica.

$F(1,0,0) = (1,2)$

$F(0,1,0) = (-1,-3)$

$F(0,0,1) = (-1,2)$

$$\sim \begin{bmatrix} 1 & 2 \\ -1 & -3 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Base $Im(F) = \{(1,2), (0,-1)\}$ dim $(F) = 2$

(É sobrejetora pois todo conjunto imagem esta contido no $\mathbb{R}^2$

$F: U \to V$, quando $U = V$, isto é um operador linear

Dada a TL $F: \mathbb{R}^3 \to \mathbb{R}^3$ / $F(x,y,z) = (x, x-y, x-y-z)$, determinar

a) uma base do Nuc(F) e dim (F)

b) uma base da Im (F) e dim Im(F)

a) definição de base do núcleo

$F(u) = 0$

$F(0,0,0) = (0,0,0)$

$(0,0,0) = (x, x-y, x-y-z)$

$$\begin{cases} x = 0 \\ x - y = 0 \quad \therefore \quad x = y = 0 \\ x - y - z = 0 \quad \therefore \quad z = 0 \end{cases}$$

$(x,y,z) \in F(u)$ /

$(x,y,z) = (0,0,0)$

$\therefore$ Base do Nucleo(F) = { }

$\dim(F) = 0$

$\therefore$ F é injetora, pois o Nucleo(F) = { }

b) $\operatorname{Im} f = \{u \in U \mid \exists v \in V \text{ e } v = F(u)\}$

$F(1,0,0) = (1,1,1)$

$F(0,1,0) = (0,1,-1)$

$F(0,0,1) = (0,0,-1)$

Como os vetores são LI

$\therefore$ Base Imf(F) = {(1,1,1), (0,1,-1), (0,0,-1)}

dim Imf(F) = 3

F é injetora, pois a base do Núcleo = { } e é sobrejotora pois a Im F $\in$ V. Portanto é automorfismo, pois a dimensão do domínio é igual a dimensão do contradomínio.

$f: \mathbb{R} \to \mathbb{R}$

$x \mapsto y = f(x) = x^2$ é linear?

a) $F(u_1 + u_2) = F(u_1) + F(u_2)$

b) $F(\lambda u) = \lambda(F(u))$

$F(u_1 + u_2) = (u_1 + u_2)^2 = u_1^2 + 2u_1u_2 + u_2^2$

$F(u_1) + F(u_2) = u_1^2 + u_2^2$

Como $F(u_1 + u_2) \neq F(u_1) + F(u_2)$ então não é linear

1-) T.L: $T: R^3 \to R^4$

$T(x,y,z) = (x+y+z,\ y+z,\ 3x+y,\ 2y+z)$

a) Base do ker (T)

$F(u) = 0$

$F(x,y,z) = (0,0,0,0)$

$(0,0,0,0) = (x+y+z,\ y+z,\ 3x+y,\ 2y+z)$

$$\begin{cases} x+y+z=0 \\ y+z=0 \\ 3x+y=0 \\ 2y+z=0 \end{cases} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x+y+z=0 \\ y+z=0 \\ -z=0 \end{cases} \quad \therefore x = y = z = 0$$

Base do ker $\{\ \}$ dim Ker (T) = 0

b) $Im\,F = \{ u \in U \mid v \in V \text{ e } v = F(u) \}$

$T(1,0,0) = (1,0,3,0)$
$T(0,1,0) = (1,1,1,2)$
$T(0,0,1) = (1,1,0,1)$

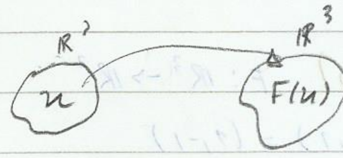
$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 1 & -3 & 1 \end{bmatrix}$$

Base $Im(T) = \{(1,0,3,0), (0,1,-3,1), (0,0,-2,2)\}$

Como a dimensão da Imagem não completa o contradomínio, então não é sobrejetora. Portanto não é bijetora, (que não pode ser isomorfismo e automorfismo)

3-) ~~$T: P_4(x) \to P_4(x)$ $T(p(x)) = \frac{d^2p(x)}{dx^2} + 3\frac{dp(x)}{dx}$. Justificar que é transformação linear~~

$F(1,0,0) = (1,1,0)$ $F: \mathbb{R}^3 \to \mathbb{R}^3$
$F(0,2,0) = (2,0,6)$
$F(0,1,-1) = (2,-2,3)$



$(x,y,z) = a(1,0,0) + b(0,2,0) + c(0,1,-1)$

$$\begin{cases} a = x & a = x \\ 2b + c = y & b = \frac{y - c}{2} \therefore b = \frac{y+z}{2} \\ -c = z & c = -z \end{cases}$$

$F(x,y,z) = a\,F(1,0,0) + b\,F(0,2,0) + c\,F(0,1,-1)$
$= x(1,1,0) + \frac{y+z}{2}(2,0,6) - z(2,-2,3)$

$\therefore F(x,y,z) = (x + y + \cancel{z} - \cancel{z},\ x + 2z,\ 3y + \cancel{3z} - \cancel{3z})$

129-) (Livro) $T(1,0,1) = (-1,-1,0)$ $\quad T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(0,1,-1) = (-1,1,2)$

$T(0,0,-1) = (0,-1,3)$

$(x,y,z) = a(1,0,1) + b(0,1,-1) + c(0,0,-1)$

$$\begin{cases} a = x \\ b = y \\ a - b - c = z \end{cases} \quad c = a - b - z = x - y - z$$

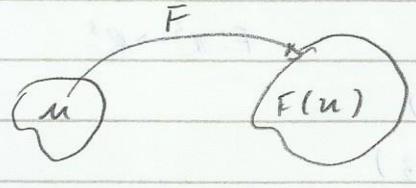
$T(x,y,z) = a\,T(1,0,1) + b\,T(0,1,-1) + c\,T(0,0,-1)$

$= x(-1,-1,0) + y(-1,1,2) + (x-y-z)(0,-1,3)$

$T(x,y,z) = (-x-y, -x+y-x+y+z, 2y+3x-3y-3z)$

$\therefore T(x,y,z) = (-x-y, -2x+2y+z, 3x-y-3z)$

132-) (Livro) $F: \mathbb{R}^3 \to \mathbb{R}^2$



$F(1,0,1) = (1,-1)$

$F(0,1,1) = (2,3)$

$F(0,0,1) = (0,2)$

$(x,y,z) \in \mathbb{R}^3$

$(x,y,z) = a(1,0,1) + b(0,1,1) + c(0,0,1)$

$$\begin{cases} a = x \\ b = y \\ a + b + c = z \end{cases} \quad c = z - a - b \;\therefore\; c = z - x - y$$

$F(x,y,z) = a\,F(1,0,1) + b\,F(0,1,1) + c\,F(0,0,1)$

$F(x,y,z) = x(1,-1) + y(2,3) + (-x-y+z)(0,2)$

$= (x+2y, -3x+y+2z)$

139-) $F: \mathbb{R}^3 \to \mathbb{R}^4 \mid F(x,y,z) = (x+2z,\ x+2y,\ y-z,\ 2x+4z)$

$G: \mathbb{R}^4 \to \mathbb{R}^4 \mid G(x,y,z,t) = (x-y+t,\ t-y,\ z+t,\ 2x-2y+2t)$

Im(F):

$F(1,0,0) = (1,1,0,2)$

$F(0,1,0) = (0,2,1,0)$

$F(0,0,1) = (2,0,-1,4)$

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Base $Im(F) = \{(1,1,0,2),(0,2,1,0)\}$

$\dim Im(F) = 2$

Im(G)

$G(1,0,0,0) = (1,0,0,2)$

$G(0,1,0,0) = (-1,-1,0,-2)$

$G(0,0,1,0) = (0,0,1,0)$

$G(0,0,0,1) = (1,1,1,2)$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Base $Im(G)$

$= \{(1,0,0,2),(0,-1,0,0),(0,0,1,0)\}$

$\dim Im(G) = 3$

$Im(F) + Im(G)$ $\quad \{(1,0,0,2),(0,-1,0,0),(0,0,1,0)\}$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

142-) $T: \mathbb{R}^3 \to \mathbb{R}^3$: $T(x,y,z) = (x-y+2z,\ 2x+y,\ -x-2y+2z)$

$v=(a,b,c)$ pertencer a imagem de T ?

$Im(T) = \{v \in V \mid \exists u \in U \text{ e } v = F(u)\}$

$(a,b,c) = (x-y+2z,\ 2x+y,\ -x-2y+2z)$

$$\begin{cases} x-y+2z = a \\ 2x+y = b \\ -x-2y+2z = c \end{cases} \sim \begin{bmatrix} 1 & -1 & 2 & a \\ 2 & 1 & 0 & b \\ -1 & -2 & 2 & c \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 3 & -4 & -2a+b \\ 0 & -3 & 4 & a+c \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 3 & -4 & -2a+b \\ 0 & 0 & 0 & -a+b+c \end{bmatrix}$$

$\underline{-a+b+c=0}$

$\therefore \underline{a=b+c}$

143) $L: P_3(R) \to P_3(R)$

$L(at^3+bt^2+ct+d) = (a-b)t^3 + (c-d)t$

a) Base do $\ker(F)$

$\ker(F)$: $F(u) = 0$

$$\begin{cases} a-b=0 & a=b \\ c-d=0 & c=d \end{cases}$$

$(at^3+bt^2+ct+d) = (at^3+at^2+ct+c)$

$a(t^3+t^2+0+0) + c(0+0+t+1)$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Base do ker $\{t^3+t^2;\ t+1\}$

Dimensão do ker = 2

$Im(F) = \{v \in V \mid u \in V \text{ e } v = F(u)\}$

Base canônica $P_3(\mathbb{R}) = \{1, t, t^2, t^3\}$

$L(1) = -t \quad L(t) = t \quad L(t^2) = -t^3 \quad L(t^3) = t^3$

$B = \{-t, t, -t^3, t^3\}$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$B = \{-t, -t^3\}$

$Dim\ Im(F) = 2$

$Dim\ V = Dim\ Ker + Dim\ Im(F)$

$4 = 2 \quad 2$ ✓

144-) $F: M_2(\mathbb{R}) \to M_3(\mathbb{R})$

$$F = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & a-b & c \\ b-a & 0 & d-c \\ -c & c-d & 0 \end{bmatrix}$$

$F(u) = 0$

$$\begin{bmatrix} 0 & a-b & c \\ b-a & 0 & d-c \\ -c & c-d & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a = b \quad d = c \quad \therefore d = 0$

$b = a \quad c = 0$

/ /

$$F\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ 0 & 0 \end{bmatrix} = b\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$a = 0$

$b = 0 \quad \therefore \ l_i$

Transformação linear inversa

$F: \mathbb{R}^3 \to \mathbb{R}^3$

$$F(x,y,z) = (x,\ x+y,\ x+y+z)$$

$F^{-1}(x,y,z) = (a,b,c)$

$F(a,b,c) = (x,y,z) = (a,\ a+b,\ a+b+c)$

145-) $F: \mathbb{R}^2 \to \mathbb{R}^2$

$F(1,1) = (1,2)$

$F(2,0) = (4,2)$

$(x,y) = a(1,1) + b(2,0)$

$$\begin{cases} a + 2b = x \\ a = y \end{cases} \quad \therefore \ y + 2b = x \quad \therefore \ b = \frac{x-y}{2}$$

$F(x,y) = a\,F(1,1) + b\,F(2,0)$

$\qquad = y(1,2) + \frac{x-y}{2}(4,2)$

$\therefore \ F(x,y) = (y + 2x - 2y,\ 2y + x - y)$

$\therefore \ F(x,y) = (2x - y,\ x + y)$

$F^{-1}(x,y) = (a,b) \iff F(a,b) = (x,y)$

$(2a-b\ ,\ a+b) = (x,y)$

$\begin{cases} 2a-b = x \\ a+b = y \end{cases} \sim \begin{cases} 2a-b = x \\ 3a = x+y \end{cases} \quad \therefore a = \dfrac{x+y}{3}$

$b = 2a - x = \dfrac{2}{3}(x+y) - x = \dfrac{2x+2y-3x}{3}$

$\therefore b = \dfrac{-x+2y}{3}$

$\therefore F^{-1}(x,y) = (a,b)$

$F^{-1}(x,y) = \left(\dfrac{x+y}{3}\ ,\ \dfrac{-x+2y}{3} \right)$

145-) $F: \mathbb{R}^2 \to \mathbb{R}^2$

$F(1,1) = (1,2) \qquad F(x,y) = ?$

$F(2,0) = (4,2) \qquad F^{-1}(x,y) = ?$

$(x,y) = a(1,1) + b(2,0)$

$\begin{cases} a+2b = x \\ a = y \end{cases} \quad b = \dfrac{x-y}{2} \quad a = y$

$F(x,y) = aF(1,1) + bF(2,0)$

$F(x,y) = y(1,2) + \dfrac{x-y}{2}(4,2) \quad \therefore F(x,y) = (\underset{a}{-y+2x}\ ,\ \underset{b}{x+y})$

$F^{-1}(x,y) = (a,b) \quad F(a,b) = (x,y)$

$(2a-b\ ,\ a+b) = (x,y) \qquad \therefore F^{-1}(x,y) = \left(\dfrac{x+y}{3}\ ,\ \dfrac{2y-x}{3} \right)$

$\begin{cases} 2a-b = x \\ a+b = y \end{cases} \quad \therefore a = \dfrac{x+y}{3} \quad b = \dfrac{2y-x}{3}$

/ /

146) $F: \mathbb{R}^3 \to \mathbb{R}^3$: $F(x,y,z) = (x+z, y-2z, x+y)$

Para F ser automorfismo a base do ker { } e a dimensão da imagem tem que ser igual ao do contradomínio, para ser sobrejetora, e a dimensão do domínio e contradomínio tem que ser igual.

$$F(u) = (0)$$
$$F(x,y,z) = (0,0,0)$$

$$(0,0,0) = (x+z, y-2z, x+y)$$

$$\begin{cases} x+z=0 \\ y-2z=0 \\ x+y=0 \end{cases} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{cases} x+z=0 \\ y-2z=0 \\ -z=0 \end{cases} \therefore x=y=z=0 \therefore \text{Base do ker } \{\ \}$$

$$\begin{aligned} F(1,0,0) &= (1,0,1) \\ F(0,1,0) &= (0,1,1) \\ F(0,0,1) &= (0,-2,0) \end{aligned} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Base da $Im(F) = \{(1,0,1),(0,1,1),(0,0,2)\}$

$$\dim(F) = 3$$

$\therefore$ é automorfismo

$F(x,y,z) = (x+z,\ y-2z,\ x+y)$

$F^{-1}(x,y,z) = (a,b,c) \quad \therefore \quad F(a,b,c) = (x,y,z)$

$(a+c,\ b-2c,\ a+b) = (x,y,z)$

$$\begin{cases} a+c = x \\ b-2c = y \\ a+b = z \end{cases} \quad \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -2 & y \\ 1 & 1 & 0 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -2 & y \\ 0 & 1 & -1 & z-x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -2 & y \\ 0 & 0 & 1 & -x-y+z \end{bmatrix} \qquad \begin{aligned} a + c &= x \\ b-2c &= y \\ c &= -x-y+z \end{aligned}$$

$b = y + 2(-x-y+z) \ \therefore\ b = -2x-y+2z$

$a = x - c = x - (-2x-y+2z)$

$\therefore a = 3x+y-2z$

$F^{-1}(x,y,z) = (3x+y-2z,\ -2x-y+2z,\ -x-y+z)$

Composição de transformação linear

$G \circ F : U \to W$

$u \mapsto (G \circ F)(u) = G(F(u))$

↳ Definição

$G \circ F: U \to W$

$u \mapsto (G \circ F)(u) = G(F(u))$

$F: \mathbb{R}^2 \to \mathbb{R}^3$

$(x,y) \mapsto F(x,y) = (x+y,\ x-y,\ 2x+3y)$

$G: \mathbb{R}^3 \to \mathbb{R}^2$

$(x,y,z) \mapsto F(x,y,z) = (x+z,\ y-z)$

$G \circ F$ e $F \circ G$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & \mathbb{R}^3 \arrow[r, "G"] & \mathbb{R}^2 \end{tikzcd}$$

Pela def $\boxed{G \circ F = G(F(u))}$

$\therefore G(F(x,y))$

$F \circ G = F(G(u))$
$= F(G(x,y,z))$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "G"] \arrow[rr, bend right, "F \circ G"'] & \mathbb{R}^2 \arrow[r, "F"] & \mathbb{R}^3 \end{tikzcd}$$

$(G \circ F) = G(F(x,y))$
$= G(x+y,\ x-y,\ 2x+3y)$
$= ((x+y)+(2x+3y),\ (x-y)-(2x+3y))$
$= (3x+4y,\ -x-4y)$

$(F \circ G) = F(G(x,y,z))$
$= F(x+z,\ y-z)$
$= (x+y)+(y-z),\ (x+y)-(y-z),\ 2(x+y)+3(y-z)$
$= (x+2y-z,\ x-z,\ 2x+5y-3z)$

Sujam:

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-d & b+c \\ b+c & a-d \end{pmatrix}$$

$$G\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-d & b-c \\ b-c & a-d \end{pmatrix}$$

a) $F \circ G = F(G(u))$

$$= F\begin{pmatrix} a-d & b-c \\ b-c & a-d \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2b-2c \\ 2b-2c & 0 \end{pmatrix}$$

b) $G \circ F = G(F(u))$

$$= G\begin{pmatrix} a-d & b+c \\ b+c & a-d \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$Im(F)$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Im(F) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$Im(F) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \text{ (Base)}$$

ker(G)

$$G\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-d & b-c \\ b-c & a-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} a = d \\ b = c \end{cases} \quad a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B\ker = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

147.)

$$F, G \in L(\mathbb{R}^2): \begin{cases} F(x,y) = (2x, y+x) \\ G(x,y) = (y, 2x) \end{cases}$$

$$\begin{aligned} F \circ G &= F(G(x,y)) \\ &= F(y, 2x) \\ &= (2y, 2x+y) \end{aligned}$$

148.) $F, G \in L(M_2(\mathbb{R}))$

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-d & b+c \\ b+c & a-d \end{pmatrix}$$

$$G\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-d & b-c \\ b-c & a-d \end{pmatrix}$$

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$$F \circ G = F(G(x))$$
$$= F\begin{pmatrix} a-d & b-c \\ b-c & a-d \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 2b-2c \\ 2b-2c & 0 \end{pmatrix}$$

149-) $F: R^2 \to R^3$ $\qquad$ $G: R^3 \to R^2$

$F(x,y) = (0, x, x-y)$ $\qquad$ $G(x,y,z) = (x-y, x+2y+3z)$

$$G \circ F \circ G = G(F \circ G) = G(F(G(x,y,z)))$$
$$= G(F(x-y, x+2y+3z))$$
$$= G(-x, 2x+3(x-y))$$
$$= G(-x, 5x-3y)$$
$$= (-x+y, 5(x-y)-3(x+2y+3z))$$
$$= (-x+y, 2x-11y-9z)$$

150-) R^2: ~~$F(x,y)=(x+y, 2x-y)$~~ $\qquad$ $F(a,b)$

$G(x,y) = (y-x, x)$

$$(F \circ G)(x,y) = F(G(x,y))$$
$$= F(y-x, x) = (y, 2y-3x)$$
$$= (y-x+x, 2(y-x)-x)$$
$$= (y, 2y-3x)$$

$$F^{-1}(y, 2y-3x) = (y-x, x)$$

151) $F: \overset{u}{\mathbb{R}^2} \to \overset{F(u)}{\mathbb{R}^2}$ $\quad F(2,1) = (1,2)$ $\quad (G \circ F) = G(F(u))$

$G: \mathbb{R}^2 \to \mathbb{R}$ $\quad G(3,6) = 7$

$$\begin{tikzcd} \overset{u}{\mathbb{R}^2} \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & \overset{F(u)}{\mathbb{R}^2} \arrow[r, "G"] & \overset{G(F(u))}{\mathbb{R}} \end{tikzcd}$$

$$(G \circ F)(4,2) = G(F(4,2))$$
$$= G(2F(2,1))$$
$$= 2G(F(2,1))$$
$$= 2G(1,2)$$
$$3(G \circ F)(4,2) = 2G(3,6)$$
$$= 2 \cdot (7)$$
$$\therefore \ (G \circ F)(4,2) = \frac{14}{3}$$

153-) $\mathbb{R}^3$: $S(x,y,z) = (z,x,y)$

$T(x,y,z) = (0, x+y+z, 0)$

a) $(S+2T)(0,1,1) = (z,x,y) + 2(0, x+y+z, 0)$
$$= (z, 3x+2y+2z, y)$$
$$= (1,4,1)$$

b) $(S \circ T)(0,1,1) = S(T(0,1,1))$
$$= S(0,2,0)$$
$$= (0,0,2)$$

c) $S^2(0,1,1) = S \circ S = S(S(0,1,1))$
$$= S(1,0,1)$$
$$= (1,1,0)$$

105-) $F: \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = (x+y, 2y)$ $\qquad G(x,y) = ?$

$(F \circ G)(x,y) = (2x-3y, x+y)$

$(F \circ G)(x,y) = F(G(x,y)) = (2x-3y, x+y)$

$= F(a, b) = (2x-3y, x+y)$

$$\begin{cases} a+b = 2x-3y \\ 2b = x+y \end{cases} \quad \therefore b = \frac{x+y}{2}$$

$a = 2x-3y-b = \frac{4x-6y-x-y}{2} \quad \therefore a = \frac{3x-7y}{2}$

$G(x,y) = \left(\frac{3x-7y}{2}, \frac{x+y}{2}\right)$

2-(Lista Base)

TL $\mathbb{R}^2 \to \mathbb{R}^2$

$S(x,y) = (\sqrt{2}x, \sqrt{2}y)$

$R(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\ \theta = 45$

$S \circ R = S(R(x,y))$

$= S(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

$= (\sqrt{2}(x\cos\theta - y\sin\theta), \sqrt{2}(x\sin\theta + y\cos\theta))$

$= (x-y, x+y)$

/ /

4-) Lista

$\{u_1 = (1,0,1);\ u_2 = (0,1,-1);\ u_3 = (1,1,-1)\}$

a) $T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(u_1) = u_2 \quad \Rightarrow \quad T(1,0,1) = (0,1,-1)$

$T(u_2) = u_3 \quad \quad T(0,1,-1) = (1,1,-1)$

$T(u_3) = v_1 \quad \quad T(1,1,-1) = (1,0,1)$

$(x,y,z) = a(1,0,1) + b(0,1,-1) + c(1,1,-1)$

$$\begin{cases} a + c = x \\ b + c = y \\ a - b - c = z \end{cases} \sim \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 1 & y \\ 1 & -1 & -1 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 1 & y \\ 0 & 1 & 2 & x-z \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 1 & x-y-z \end{pmatrix} \sim \begin{matrix} a + c = x \\ b + c = y \\ c = x-y-z \end{matrix}$$

$b = y - c = y - (x - y - z) \quad \therefore b = -x + 2y + z$

$a = x - c = x - (x - y - z) \quad \therefore a = y + z$

$F(x,y,z) = aF(1,0,1) + bF(0,1,-1) + cF(1,1,-1)$

$= (y+z)(0,1,-1) + (-x+2y+z)(1,1,-1) + (x-y-z)(1,0,1)$

$= (\cancel{-x} + 2y + z + \cancel{x} - y - z,\ y + z - x + 2y + z,\ -y - z + x - 2y - z + x - y - z)$

$\therefore F(x,y,z) = (y,\ -x + 3y + 2z,\ 2x - 4y - 3z)$

b) $F(u)=0$
$F(x,y,z)=(0,0,0)$

$(0,0,0)=(y\ ,-x+3y+2z\ ,2x-4y-3z)$

$\begin{cases} y=0 \\ -x+3y+2z=0 \\ 2x-4y-3z=0 \end{cases}$ $\begin{cases} -x+2z=0 \\ 2x-3z=0 \end{cases} \sim \begin{cases} -x+2z=0 \\ 0 \quad z=0 \end{cases}$

$\therefore\ x=y=z=0$ Base do ker $\{\ \}$

Transformação linear

Exemplo prático

$F: \mathbb{R}^3 \to \mathbb{R}^3 \ / F(x,y,z)=(x-3y+z\ ,\ 2x+y-2z)$
$\quad\ B \quad\ C$

Matriz F: em relação a base canônica

$B=\{e_1=(1,0,0);\ e_2=(0,1,0);\ e_3=(0,0,1)\} \in \mathbb{R}^3$
$C=\{e_1=(1,0)\ ,\ e_2=(0,1)\} \in \mathbb{R}^2$

$F(e_1)=F(1,0,0)=(1,2)=a(1,0)+b(0,1)$
$F(e_2)=F(0,1,0)=(-3,1)=a(1,0)+b(0,1)$
$F(e_3)=F(0,0,1)=(1,-2)=a(1,0)+b(0,1)$

$\begin{cases} a=1 \\ b=2 \end{cases}$ $\begin{cases} a=-3 \\ b=1 \end{cases}$ $\begin{cases} a=1 \\ b=-2 \end{cases}$

$(F)_{B,C}=\begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}$

/ /

2.) Matriz de F em relação a base:

$B = \{(1,0,1), (0,-1,1), (0,0,-1)\} \in \mathbb{R}^3$
$C = \{(1,1), (1,2)\} \in \mathbb{R}^2$

$F(1,0,1) = (2,0) = a_1(1,1) + b_1(1,2)$
$F(0,-1,1) = (4,-3) = a_2(1,1) + b_2(1,2)$
$F(0,0,-1) = (-1,2) = a_3(1,1) + b_3(1,2)$

$\begin{cases} a_1 + b_1 = 2 \\ a_1 + 2b_1 = 0 \end{cases} \sim \begin{cases} a_1 + b_1 = 2 \\ -b_1 = 2 \end{cases} \therefore \begin{matrix} a_1 = 4 \\ b_1 = -2 \end{matrix}$

$\begin{cases} a_2 + b_2 = 4 \\ a_2 + 2b_2 = -3 \end{cases} \sim \begin{cases} a_2 + b_2 = 4 \\ -b_2 = 7 \end{cases} \therefore \begin{matrix} a_2 = 11 \\ b_2 = -7 \end{matrix}$

$\begin{cases} a_3 + b_3 = -1 \\ a_3 + 2b_3 = 2 \end{cases} \sim \begin{cases} a_3 + b_3 = -1 \\ -b_3 = -3 \end{cases} \therefore \begin{matrix} a_3 = -4 \\ b_3 = 3 \end{matrix}$

$$(F)_{B,C} = \begin{bmatrix} 4 & 11 & -4 \\ -2 & -7 & 3 \end{bmatrix}$$

Seja $F: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad F(x,y) = (3x - y, x - 2y)$
$B = \{u_1 = (1,0)\,;\, u_2 = (0,1)\}$ $\quad C = \{v_1 = (1,0)\,;\, v_2 = (0,1)\}$

$F(u_1) = (3,1) = a_1(1,0) + b_1(0,1)$
$F(u_2) = (-1,-2) = a_2(1,0) + b_2(0,1)$

$a_1 = 3$ $\quad a_2 = -1$
$b_1 = 1$ $\quad b_2 = -2$

$$(F)_{B,C} = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$$

TOT

166-) $T: P_2(R) \to P_2(R)$ $\qquad C = \{1, x, x^2\}$

$T(p(x)) = (1-2x).p'(x)$

$T(1) = (1-2x).0 = 0.1 + 0x + 0x^2$

$T(x) = (1-2x).1 = 1.1 - 2x + 0x^2$

$T(x^2) = (1-2x).2x = 0.1 + 2x - 4x^2$

$$M = (T)_C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -4 \end{pmatrix}$$

167-) $(T) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$

$T(1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1) = (1,0,0)$

$T(0,1,0) = -1(1,0,0) + 1(0,1,0) + 2(0,0,1) = (-1,1,2)$

$T(0,0,1) = 0(1,0,0) - 1(0,1,0) + 2(0,-1,2) = (0,-1,2)$

$(x,y,z) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$a = x \quad ; \quad b = y \quad , \quad c = z$

$T(x,y,z) = aT(1,0,0) + bT(0,1,0) + cT(0,0,1)$

$= x(1,0,0) + y(-1,1,2) + z(0,-1,2)$

$T(x,y,z) = (x-y, y-z, 2y+2z)$

$T^2 = T \circ T = T(T(x,y,z)) = T(x-y, y-z, 2y+2z)$

$= (x-y-y+z, y-z-2y-2z, 2y-2z+4y+4z)$

$= (x-2y+z, -y-3z, 6y+2z)$

/ /

$F = (x,y,z) + (x-y, y-z, 2y+2z) + (x-2y+z, -y-3z, 6y+2z)$

$\therefore F = (3x-3y+z, y-4z, 8y+5z)$

$F(1,0,0) = (3,0,0) = a_1(1,0,0) + b_1(0,1,0) + c_1(0,0,1)$
$F(0,1,0) = (-3,1,8) = a_2(1,0,0) + b_2(0,1,0) + c_2(0,0,1)$
$F(0,0,1) = (1,-4,5) = a_3(1,0,0) + b_3(0,1,0) + c_3(0,0,1)$

$a_1 = 3 \quad b_1 = 0 \quad c_1 = 0$
$a_2 = -3 \quad b_2 = 1 \quad c_2 = 8$
$a_3 = 1 \quad b_3 = -4 \quad c_3 = 5$

$$(F) = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 1 & -4 \\ 0 & 8 & 5 \end{bmatrix}$$

168) $F : R^2 \to R^2$ $\quad M = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ $\quad$ Determine: $F(2,3)$

$F(1,0) = 1(1,0) - 1(0,1) = (1,-1)$
$F(0,1) = 2(1,0) + 1(0,1) = (2,1)$

$(x,y) = a(1,0) + b(0,1)$
$x = a \quad y = b$

$F(x,y) = a F(1,0) + b F(0,1)$
$= x(1,-1) + y(2,1)$

$\therefore F(x,y) = (x+2y, -x+y)$
$F(2,3) = (8,1)$

169.) $F: \mathbb{R}^3 \to \mathbb{R}^4$

$F(0,1,1) = (2,2,-1,0)$

$F(1,0,3) = (5,2,2,1)$

$F(2,1,0) = (5,-2,3,2)$

$(x,y,z) = a(0,1,1) + b(1,0,3) + c(2,1,0)$

$$\begin{cases} b + 2c = x \\ a + c = y \\ a + 3b = z \end{cases} \sim \begin{bmatrix} 0 & 1 & 2 & x \\ 1 & 0 & 1 & y \\ 1 & 3 & 0 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & y \\ 0 & 1 & 2 & x \\ 0 & 3 & -1 & z-y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & y \\ 0 & 1 & 2 & x \\ 0 & 0 & -7 & -3x-y+z \end{bmatrix} \sim \begin{cases} a + c = y \\ b + 2c = x \\ -7c = -3x - y + z \end{cases}$$

$c = \dfrac{-3x-y+z}{-7}$ $\quad b = x - 2c \quad b = x - 2 . \dfrac{(-3x-y+z)}{-7}$

$\therefore b = \dfrac{-7x+6x+2y-2z}{-7} = \dfrac{-x+2y-2z}{-7}$

$a = y - c = y - \left(\dfrac{-3x-y+z}{-7}\right) = \dfrac{-7y+3x+y-z}{-7}$

$\therefore a = \dfrac{3x-6y-z}{-7}$

$F(x,y,z) = \dfrac{3x-6y-z}{-7}(2,2,-1,0) + \dfrac{-x+2y-2z}{-7}(5,2,2,1) +$

$\dfrac{-3x-y+z}{-7}(5,-2,3,2)$

$F(x,y,z) = \dfrac{-3x+6y+z}{7}(2,2,-1,0) + \dfrac{x-2y+2z}{7}(5,2,2,1) + \dfrac{3x+y-z}{7}(5,-2,3,2)$

$F(x,y,z) = \left(2x+y+z, \dfrac{-10x+6y+6z}{7}, 2x-y, x\right)$

/ /

$F(1,0,0) = (2, -\frac{10}{7}, 2, 1) = a_1(1,0,0) + b_1(0,1,0) + c_1(0,0,1)$

$F(0,1,0) = (1, 6/7, -1, 0) = a_2(1,0,0) + b_2(0,1,0) + c_2(0,0,1)$

$F(0,0,1) = (1, 8/7, 0, 0) = a_3(1,0,0) + b_3(0,1,0) + c_3(0,0,1)$

$a_1 = 2 \quad b_1 = -10/7 \quad c_1 = 1$

$a_2 = 1 \quad b_2 = 6/7 \quad c_2 = 8/7$

$a_3 = 1 \quad b_3 = 8/7 \quad c_3 = 0$

$$(F) = \begin{pmatrix} 2 & 1 & 1 \\ -10/7 & 6/7 & 8/7 \\ 1 & 8/7 & 0 \end{pmatrix}$$

Exercício Caderno

$B = \{(1,1,1), (0,1,1), (1,0,1)\}\ \mathbb{R}^3 \qquad F: \mathbb{R}^3 \to \mathbb{R}^2$

$C = \{(1,-1), (1,2)\}\ \mathbb{R}^2$

Determinar $F(x,y,z)$ $\qquad (F)_{B,C} = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \end{pmatrix}$

$F(1,1,1) = 1(1,-1) - (1,2) = (0,-3)$

$F(0,1,1) = 0(1,-1) + 2(1,2) = (2,4)$

$F(1,0,1) = 3(1,-1) + (1,2) = (4,-1)$

$(x,y,z) = a(1,1,1) + b(0,1,1) + c(1,0,1)$

$$\begin{cases} a + c = x \\ a + b = y \\ a + b + c = z \end{cases} \sim \begin{bmatrix} 1 & 0 & 1 & x \\ 1 & 1 & 0 & y \\ 1 & 1 & 1 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -1 & y-x \\ 0 & 1 & 0 & z-x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -1 & y-x \\ 0 & 0 & 1 & z-y \end{bmatrix} \sim \begin{cases} a + c = x \\ b - c = y - x \\ c = z - y \end{cases}$$

$b = c + y - x$
$= z - y + y - x \Rightarrow b = -x + z$

$a = x - c = x - (z - y)$
$\therefore a = x - z - y$

$F(x,y,z) = aF(1,1,1) + bF(0,1,1) + cF(1,0,1)$
$= (x - y - z)(0,-3) + (-x+z)(2,1) + (z-y)(4,-1)$
$= (-2x + 2z + 4z - 4y, -3x + 3y + 3z - 4x + 4z - z + y)$

$\therefore F(x,y,z) = (-2x - 4y + 6z, -7x + 4y + 6z)$

170-) $F: \mathbb{R}^3 \to M_2(\mathbb{R})$
$F(x,y,z) = \begin{pmatrix} x-y & y-z \\ z-y & x-z \end{pmatrix}$

Determine uma base do núcleo e da imagem F.

Para determinar a base do ker $F(u) = 0$
$\therefore F(x,y,z) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} x-y & y-z \\ z-y & x-z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad x - y = 0 \quad x = y$
$\quad z - y = 0 \quad z = y$

$y - z = 0 \quad y = z \qquad$ se $x = z = y$
$x - z = 0 \quad x = z$

$(x,y,z) = (x,x,x)$
$= x(1,1,1)$

Base do ker $\{(1,1,1)\}$

/ /

$F(1,0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $F(0,1,0) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ $F(0,0,1) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

171) $T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(x,y,z) = (x-2y+z, 2x+y-3z)$

$B = \{(1,1,0); (-1,0,2); (0,0,1)\}$

$C = \{(1,-1); (0,1)\}$

$T(1,1,0) = (-1,3) = a_1(1,-1) + b_1(0,1)$

$T(-1,0,2) = (1,-6) = a_2(1,-1) + b_2(0,1)$

$T(0,0,1) = (1,-3) = a_3(1,-1) + b_3(0,1)$

$\begin{cases} a_1 = -1 \\ -a_1 + b_1 = 3 \end{cases}$ $\therefore b_1 = 3 + a_1 = 3 - 1 \therefore b_1 = 2$

$\begin{cases} a_2 = 1 \\ -a_2 + b_2 = -6 \end{cases}$ $b_2 = -6 + a_2 \therefore b_2 = -7$

$\begin{cases} a_3 = 1 \\ -a_3 + b_3 = -3 \end{cases}$ $\therefore b_3 = -3 + a_3 \therefore b_3 = -3 + 1 \Rightarrow b_3 = -2$

$(T)_{B,C} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 7 & -2 \end{bmatrix}$

172-) $F: \mathbb{R}^3 \to \mathbb{R}^2$

$B = \{(1,1,1), (1,1,0), (1,0,0)\}$ do $\mathbb{R}^3$

$C = \{(1,1), (1,0)\}$ do $\mathbb{R}^2$

$$(T)_{B,C} = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$F(1,1,1) = 3(1,1) + 1(1,0) = (4,3)$

$F(1,1,0) = 5(1,1) + 2(1,0) = (7,5)$

$F(1,0,0) = 1(1,1) + 0(1,0) = (1,1)$

$(x,y,z) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$$\begin{cases} a+b+c = x & c = x-a-b \quad \therefore \quad c = x - z - y + z \quad \therefore c = x - y \\ a+b = y & b = y - a \quad \therefore \quad b = y - z \\ a = z \end{cases}$$

$F(x,y,z) = aF(1,1,1) + bF(1,1,0) + cF(1,0,0)$

$= z(4,3) + (y-z)(7,5) + (x-y)(1,1)$

$= (4z + 7y - 7z + x - y,\ 3z + 5y - 5z + x - y)$

$F(x,y,z) = (x + 6y - 3z,\ x + 4y - 2z)$

$F(1,0,0) = (1,1)$

$F(0,1,0) = (6,4)$ $\quad (F) = \begin{bmatrix} 1 & 6 & -3 \\ 1 & 4 & -2 \end{bmatrix}$

$F(0,0,1) = (-3,-2)$

173-) $F: \mathbb{R}^3 \to \mathbb{R}^2$

$F(x,y,z) = (3x+2y-4z\ ,\ x-5y+3z)$

$B = \{(1,1,1); (1,1,0); (1,0,0)\}$ do $\mathbb{R}^3$

$C = \{(1,3); (2,5)\}$ do $\mathbb{R}^2$

$F(1,1,1) = (1,-1) = a_1(1,3) + b_1(2,5)$

$F(1,1,0) = (5,-4) = a_2(1,3) + b_2(2,5)$

$F(1,0,0) = (3,1) = a_3(1,3) + b_3(2,5)$

$\begin{cases} a_1 + 2b_1 = 1 \\ 3a_1 + 5b_1 = -1 \end{cases} \sim \begin{cases} a_1 + 2b_1 = 1 \\ -b_1 = -4 \end{cases} \quad a = 1 - 2b_1 \ \therefore a_1 = -7 \quad \therefore b_1 = 4$

$\begin{cases} a_2 + 2b_2 = 5 \\ 3a_2 + 5b_2 = -4 \end{cases} \sim \begin{cases} a_2 + 2b_2 = 5 \\ -b_2 = -19 \end{cases} \quad a_2 = 5 - 2b_2 \ \therefore a_2 = -33 \quad \therefore b_2 = 19$

$\begin{cases} a_3 + 2b_3 = 3 \\ 3a_3 + 2b_3 = 1 \end{cases} \sim \begin{cases} a_3 + 2b_3 = 3 \\ -b_3 = -8 \end{cases} \quad a_3 = 3 - 2b_3 \ \therefore a_3 = -13 \quad \therefore b_3 = 8$

$\therefore (F)_{B,C} = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$

174-) $F: \mathbb{R}^3 \to \mathbb{R}^3$

$F(x,y,z) = (x-2z\ ,\ x+2y-z\ ,\ x-y+3z)$

a) base e dimensão do núcleo

b) " " " da imagem

c) representação matricial do operador F

a) $F(u) = 0$

$F(x,y,z) = (0,0,0)$

$(0,0,0) = (x - 2z,\ x + 2y - z,\ x - y + 3z)$

$$\begin{cases} x - 2z = 0 \\ x + 2y - z = 0 \\ x - y + 3z = 0 \end{cases} \sim \begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 11 \end{bmatrix}$$

$$\begin{cases} x - 2z = 0 \\ 2y + z = 0 \\ 11z = 0 \end{cases} \quad \therefore x = y = z = 0$$

$(x,y,z) \in Ker \,/$

$(x,y,z) = (0,0,0)$ $\therefore$ Base do ker $\{\ \}$ ou $\{0\}$

dimensão do ker $= 0$

b) $F(1,0,0) = (1,1,1)$

$F(0,1,0) = (0,2,-1)$

$F(0,0,1) = (-2,-1,3)$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ -2 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -11 \end{bmatrix}$$

Base da $Im F = \{(1,1,1), (0,2,-1), (0,0,-1)\}$

$\dim Im F = 3$

$\dim V = \dim Im F + \dim Nuc(F)$

$3 = 3 + 0$

/ /

c) $F(1,0,0)=(1,1,1)$
$F(0,1,0)=(0,2,-1)$
$F(0,0,1)=(-2,-1,3)$

$$(F)=\begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Recapitulação do que foi estudado

$F: U \rightarrow V$
↳ contradomínio
↳ domínio
↳ lei ou aplicação

Para provar que é uma transformação linear. Devemos:
i) $F(u_1+u_2)=F(u_1)+F(u_2)$
ii) $F(\lambda u)=\lambda F(u)$

Para determinar a base do ker, é só igualar o operador igual a $\emptyset$:

$F(u)=0$ Exemplo: $F(x,y)=(0,0)$
$F(t^2+t)=(0t^2+0t)$
$F\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Dica de resolução
$F(x,y)=(0,0)$
$(x,y)\in Nuc(F)\,|$
$(x,y)=(x,x)=x(1,1)$
} Exemplo

Para determinar a imagem, devemos utilizar os vetores da base canônica, quando efetuados os cálculos, deve escalonar para verificar a dependência linear

- A base é injetora, quando o nucleo for conjunto vazio, e $x_1\neq x_2$ $f(x_1)\neq f(x_2)$

Para a base ser sobrejetora, a dimensão da base da imagem deve ser igual a dimensão do contradomínio.

Caso a base seja, injetora e sobrejetora, ela é uma base bijetora, se a dimensão do contradomínio for igual a dimensão do domínio. ela é automorfismo, caso contrário isomorfismo.

A base só tem inversa se for bijetora.

Composição de Transformação linear

$G \circ F = G(F(u))$
$F \circ G = F(G(u))$
$I(u) = u \quad , \forall u \in U$
$(I \circ F) = (F \circ I) = F$
$G^2 = G \circ G$

Exercícios de Fixação

159-) $F(x,y) = (x+y, x-y)$
$G(x,y) = (-y, x)$

$F^2(x,y) = F \circ F = F(F(x,y))$
$= F(x+y, x-y)$
$= (x+y+x-y, x+y-x+y)$

$\therefore F^2(x,y) = (2x, 2y)$

B) $(G+F)(x,y) = (x+y, x-y) + (-y, x)$
$= (x, 2x-y)$

c) $(F \circ G)(x,y) = F(G(x,y))$
$= F(-y, x)$
$= (-y+x, -y-x)$

$(G \circ F)(x,y) = G(F(x,y))$
$= G(x+y, x-y)$
$= (-x+y, x+y)$

153.) $S(x,y,z) = (z, x, y)$
$T(x,y,z) = (0, x+y+z, 0)$

$(S+2T)(0,1,1) = (z, x, y) + 2(0, x+y+z, 0)$
$= (z, 3x+2y+2z, y)$
$= (1, 4, 1)$

$(S \circ T)(0,1,1) = S(T(x,y,z))$
$= S(0, x+y+z, 0)$
$= (0, 0, x+y+z)$
$= (0, 0, 2)$

$S^2 = S \circ S = S(S(x,y))$
$\overset{(0,1,1)}{=} S(z, x, y)$
$= S(y, z, x)$
$= (1, 1, 0)$

182-) $\begin{cases} F(1,0,0) = (1,1,1) \\ F(0,1,0) = (1,0,1) \\ F(0,0,1) = (0,0,4) \end{cases}$

a)

$(x,y,z) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$a = x \quad b = y \quad z = z$

$F(x,y,z) = aF(1,0,0) + bF(0,1,0) + cF(0,0,1)$

$F(x,y,z) = x(1,1,1) + y(1,0,1) + z(0,0,4)$

$F(x,y,z) = (x+y, x, x+y+4z)$

b) $(F+I)(x,y,z) = (x+y, x, x+y+4z) + (x,y,z)$

$= (2x+y, x+y, x+y+5z)$

c) $F(1,0,0) = (2,1,1)$

$F(0,1,0) = (1,1,1)$

$F(0,0,1) = (0,0,5)$

$(F+I)_{B,C} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix}$

183-) $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & -3 \\ -1 & 2 & -4 \end{bmatrix}$

$(x,y,z) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$a = x \quad b = y \quad c = z$

$F(x,y,z) = aF(1,0,0) + bF(0,1,0) + cF(0,0,1)$

$= x(2,0,-1) + y(1,3,2) + z(3,-3,-4)$

$F(x,y,z) = (2x+y+3z, 3y-3z, -x+2y-4z)$

$F(1,0,0) = (2,0,-1)$

$F(0,1,0) = (1,3,2)$

$F(0,0,1) = (3,-3,-4)$

Base do Núcleo $F(x,y,z) = 0$

$(2x+y+3z, 3y-3z, -x+2y-4z) = (0,0,0)$

$\begin{cases} 2x+y+3z = 0 \\ 3y-3z = 0 \\ -x+2y-4z = 0 \end{cases} \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & -3 \\ -1 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & -3 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

/ /

$$\begin{cases} 2x + y + 3z = 0 \\ 3y - 3z = 0 \end{cases} \quad 2x = -y - 3 \quad \therefore 2x = -4y \therefore x = -2y$$
$$y = z$$

$(x, y, z) \in$ Nucleo /
$(x, y, z) = (-2y, y, y)$
$= y(-2, 1, 1)$ $\therefore$ Base do Nucl(F) $= \{(-2, 1, 1)\}$
dim Nucl(F) = 1

$F(1,0,0) = (2,0,-1)$
$F(0,1,0) = (1,3,2)$
$F(0,0,1) = (3,-3,-4)$

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & -6 & -5 \\ 0 & -6 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & -6 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Base Im f $= \{(2,0,-1), (0,-6,5)\}$

dim V = dim Im f + dim Ker(F)
3 = 1 2 ✓

Espaços Vetoriais Euclidianos - Produto Interno

Produto Interno é uma aplicação que associa a cada par ordenado de vetores $u, v \in V$ um único número real

indicação de um produto interno $\langle u, v \rangle$

a) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
b) $\langle u, v \rangle = \langle v, u \rangle$
c) $\langle \alpha v, v \rangle = \alpha \langle u, v \rangle$
d) $\langle u, u \rangle \geq 0$ se $v \neq 0$

Produtos Internos Usuais

a) No espaço vetorial $V = \mathbb{R}^n$

$u = (x_1, x_2, \cdots, x_n)$

$v = (y_1, y_2, \cdots, y_n)$

$\langle u, v \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

b) No espaço vetorial $P_n(\mathbb{R})$

$p(x)$ e $q(x) \in V$

$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)\,dx$

Obs.: Caso o exercício não conter os limite da Integral. $C(1,2)$, por exemplo, deve-se utilizar o $(0$ a $1)$.

b) No espaço vetorial $M_{n \times n}(\mathbb{R})$

$A, B \in V$

$\langle A, B \rangle = \underbrace{tr(B^tA)}$

↳ Soma dos elementos da diagonal principal

Norma de um vetor

Obs.: Um espaço vetorial V, munido de um produto interno é denominado espaço vetorial euclidiano

Seja V um espaço vetorial euclidiano e $u \in V$. A norma de um vetor u, denotada por $\|u\|$, é definida por $\|u\| = \sqrt{\langle u, v \rangle}$

Exercícios de Fixação

1) Sejam $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ $B = \begin{pmatrix} -1 & 2 & 3 \\ 3 & 1 & -2 \end{pmatrix}$

Calcule $\langle A, B \rangle$

$\langle A, B \rangle \; tr(B^t A)$

$B^t A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} -1 & 2 & 3 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 4 & \\ & & -7 \end{bmatrix}$

$\langle A, B \rangle = tr(B^t A) = 1 + 4 - 7 = \underline{-2}$

2) $V = P_2(\mathbb{R})$

$p(t) = t^2 + t$

$q(t) = t - 1$

$$\begin{aligned} \langle p(t), q(t) \rangle & \int_0^1 p(t)\, q(t)\, dt \\ &= \int_0^1 (t^2 + t)(t-1)\, dt \\ &= \int_0^1 t^3 - t^2 + t^2 - t\, dt \\ &= \int_0^1 t^3 - t\, dt \\ &= \frac{1}{4} t^4 - \frac{1}{2} t^2 \Big|_0^1 = \frac{1}{4} - \frac{1}{2} = \frac{1-2}{4} = \frac{-1}{4} \end{aligned}$$

$\langle p(t), q(t) \rangle = -\frac{1}{4}$

3) Seja $V = \mathbb{R}^4$

$u = (1,1,1,-1)$ Determine $\langle u, v \rangle$

$v = (1,2,1,1)$

$\langle u, v \rangle = 1 + 2 + \cancel{1} - \cancel{1} = 3$

Distância entre dois vetores

Seja V um espaço vetorial euclidiano, em que foi definido um produto interno $\langle u, v \rangle$. Dados dois vetores quaisquer $u, v \in V$ a distância entre esses dois vetores, que se indica $d(u,v)$ é o número real positivo $d(u,v) = \|u - v\|$

Exercício Fixação

$V = P_2(\mathbb{R})$ $p(t) = t^2 + t$

$q(t) = t - 1$

a) $\|p\| = \sqrt{\langle p(t), p(t) \rangle}$

$$\langle p(t), p(t) \rangle = \int_0^1 (t^2 + t)^2 \, dt$$
$$= \int_0^1 t^4 + 2t^3 + t^2 \, dt$$
$$= \frac{1}{5} t^5 + \frac{1}{2} t^4 + \frac{1}{3} t^3 \Big|_0^1$$
$$= \frac{1}{5} + \frac{1}{2} + \frac{1}{3} = \frac{6 + 15 + 10}{30} = \frac{31}{30}$$

$\|p\| = \sqrt{\frac{31}{30}}$

/ /

b) $d(p,q) = \|p(t) - q(t)\| = \sqrt{<p(t)-q(t) ; p(t)-q(t)>}$

$<p(t)-q(t); p(t)-q(t)> \int_0^1 (t^2+t-t+1)^2\, dt$

$= \int_0^1 (t^2+1)^2\, dt$

$= \int_0^1 t^4 + 2t^2 + 1\, dt$

$= \frac{1}{5}t^5 + \frac{2}{3}t^3 + t \Big|_0^1 = \frac{3+10+15}{15} = \frac{28}{15}$

$d(p,q) = \sqrt{\frac{28}{15}}$

c) $<p,q> = \int_0^1 (t^2+t)(t-1)\, dt$

$= \int_0^1 t^3 - t^2 + t^2 - t\, dt$

$= \int_0^1 t^3 - t\, dt$

$= \frac{1}{4}t^4 - \frac{1}{2}t^2 \Big|_0^1 = -\frac{1}{4}$

Como o $<p,q> \neq 0$, $p(t)$ e $q(t)$ não são ortogonais

209) $V = \mathbb{R}^3$

$u = (-1,1,0)$

$v = (0,1,2)$

a) $w \in V$, que seja ortogonal a u e v

$\|w\| = 9$

$(-1,1,0).(a,b,c) = 0$

$-a + b = 0 \quad a = b = -2c$

$(0,1,2).(a,b,c) = 0$

$b + 2c = 0 \rightarrow b = -2c$

$a^2 + b^2 + c^2 = 81$

$a^2 + b^2 + c^2 = 81$

$4c^2 + 4c^2 + c^2 = 81$

$9c^2 = 81$

$c^2 = 9$

$c = \pm 3$

$\pm(-6,-6,3)$

b) $\langle t, u\rangle = 10$

$\langle t + u, v\rangle = -5$ $\qquad\qquad$ R: $t(-16-2z, -6-2z, z)$

$\langle t, v\rangle = (x, y, z)(-1, 1, 0)$

$= -x + y = 10 \Rightarrow x = y - 10 = -16 - 2z$

$\langle t+u, v\rangle = (x-1, y+1, 0)(0, 1, 2)$

$= y + 1 = -5$

$\langle t, v\rangle + \langle u, v\rangle = -5$

$t = (-16, -6, z)$ $\qquad$ $\langle t, v\rangle = y + 2z$

$\langle u, v\rangle = 1$

210) $p(t) = t$ $\qquad$ $\langle t+u, v\rangle = y + 2z + 1 = -5$

$q(t) = 1 - t^2$ $\qquad$ $y = -6 - 2z$

a) $\langle p, q\rangle = \int_0^1 t(1-t^2)\,dt$

$= \int_0^1 t - t^3\,dt$

$= \frac{1}{2}t^2 - \frac{1}{4}t^4 \Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4} = \frac{1}{4}$

b) $\|q\| = \sqrt{\langle q(t), q(t)\rangle}$

$\langle q(t), q(t)\rangle = \int_0^1 (1-t^2)^2\,dt$

$= \int_0^1 1 - 2t^2 + t^4\,dt$

$= t - \frac{2}{3}t^3 + \frac{1}{5}t^5 \Big|_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$

$\|q\| = \sqrt{\frac{8}{15}}$

/ /

$(3x+2)(5x^2+x+2)$

$15x^3+3x^2+6x+10x^2+2x+4$

$\int_{-1}^{1} 15x^3+13x^2+6x+4\,dx$

$\frac{15}{4}x^4+\frac{13}{3}x^3+4x^2+4x\Big|_{-1}^{1}$

$\cancel{\frac{15}{4}}+\frac{13}{3}+\cancel{4}+4-\left(\cancel{\frac{15}{4}}-\frac{13}{3}\cancel{+4}-4\right)$

$\frac{26}{3}+8\quad\frac{26+24}{3}=\frac{40}{3}$

$(G\circ F)_{(x,y)}=G(F(x,y))=(2x-3y,-2y+3x)$

$=G(a,b)=(2x-3y,-2y+3x)$

$\begin{cases}2x-3y=a\\-2y+3x=b\end{cases}\quad\begin{cases}2x-3y=a\\3x-2y=b\end{cases}\sim\begin{cases}-6x+9y=-3a\\6x-4y=2b\end{cases}$

$\begin{cases}2x-3y=a\\5y=-3a+2b\end{cases}$

Exercício 1 (Prova Dircu)

1) $V=C[-1,1]$

p.i $\langle f(x),g(x)\rangle\ \int_{-1}^{1}f(x)g(x)dx$

$f(x)=3x+2\quad g(x)=5x^2+x+\alpha$

$\langle f(x),g(x)\rangle=\int_{-1}^{1}(3x+2)(5x^2+x+\alpha)\,dx$

$=\int_{-1}^{1}15x^3+3x^2+3x\alpha+10x^2+2x+2\alpha\,dx$

$=\frac{15}{4}x^4+\frac{13x^3}{3}+\frac{3x^2\alpha}{2}+\frac{\cancel{2}x^2}{\cancel{2}}+2x\alpha\Big|_{-1}^{1}$

$\frac{\cancel{15}}{\cancel{4}} + \frac{13}{3} + \frac{3\cancel{\alpha}}{\cancel{2}} + \cancel{1} + 2\alpha - \left(\frac{\cancel{15}}{\cancel{4}} - \frac{13}{3} + \frac{\cancel{3}}{\cancel{2}}\alpha + \cancel{1} - 2\alpha \right)$

$\frac{26}{3} + 9\alpha = 0 \quad \therefore \alpha = \frac{-13}{6}$

2) $F: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \longmapsto F(x,y) = (2x-3y, y+x)$ e $G\circ F$ $\mathbb{R}^2$

$(x,y) \mapsto G\circ F)(x,y) = (2x-3y, -2y+3x)$

a) $G(x,y)$. $G\circ F = G(F(x,y)) = (2x-3y, -2y+3x)$

$= G(2x-3y, y+x) = (2x-3y, -2y+3x)$

$\begin{cases} 2a-3b = 2x-3y \\ a+b = -2y+3x \end{cases}$

$\begin{bmatrix} 2 & -3 & 2x-3y \\ 1 & 1 & -2y+3x \end{bmatrix} \sim \begin{pmatrix} 2 & -3 & 2x-3y \\ 0 & -5 & -4x+y \end{pmatrix}$

$\begin{cases} 2a-3b = 2x-3y \\ -5b = -4x+y \end{cases}$ $\quad b = \frac{-4x+y}{-5} \quad a = \left(2x-3y + 3\frac{(-4x+y)}{-5} \right) \div 2$

$a = \frac{-10x+15y-12x+3y}{-10} = \frac{-22x+18y}{-10}$

$G(x,y) = \left(\frac{-22x+18y}{-10}, \frac{-4x+5}{-5} \right)$

$G(2x-3y, y+x)$

$\frac{-22x+18}{-5} + \frac{12x-15}{-5}$

/ /

$F: \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = (x+y, 2y)$

$(F \circ G)(x,y) = (2x-3y, x+y)$ $\qquad G(x,y) = ?$

$F \circ G(x,y) = F(G(x,y)) = (2x-3y, x+y)$

$= F(a,b) = (2x-3y, x+y)$

$\begin{cases} a+b = 2x-3y \\ 2b = x+y \end{cases}$ $\qquad b = \dfrac{x+y}{2}$ $\qquad a = 2x-3y-b$

$a = 2x-3y-\dfrac{x+y}{2}$

$a = \dfrac{4x-6y-x+y}{2} = \dfrac{3x-7y}{2}$

$G(x,y) = \left(\dfrac{x+y}{2}, \dfrac{3x-7y}{2} \right)$

$F: R^2 \to R^2$

$F(x,y) = (x+y, 2x-y)$

$G(x,y) = (y-x, x)$

$(F \circ G)(x,y) = F(G(x,y))$

$= F(y-x, x)$

$= (y-x+x, 2y-2x-x)$

$= (y, -3x+2y)$

211) $V = M_{2\times3}(\mathbb{R})$

$A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 2 & 3 \\ 3 & 1 & -2 \end{pmatrix}$

a) $\langle A, B \rangle = tr(B^t A) = -3$

$$\begin{pmatrix} -1 & 3 \\ 2 & 1 \\ 3 & -2 \end{pmatrix} \times \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 4 & \\ & & -7 \end{pmatrix}$$

b) $\|A\| = \sqrt{\langle A, A \rangle}$

$\langle A, A \rangle = tr(A^t A) = 14 \qquad \therefore \|A\| = \sqrt{14}$

$$\begin{pmatrix} 2 & 1 \\ 2 & 0 \\ -1 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & & \\ & 4 & \\ & & 5 \end{pmatrix}$$

c) $\|A+B\| = \sqrt{\langle A+B, A+B \rangle}$

$\langle A+B, A+B \rangle \qquad A+B = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 0 \end{pmatrix}$

$= tr((A+B)^t (A+B)) \qquad \|A+B\| = \sqrt{38}$

$$\begin{pmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 17 & & \\ & 17 & \\ & & 4 \end{pmatrix}$$

220-) $\langle u, v \rangle = 2$ $\quad \|u\| = 1$

$\langle v, w \rangle = 3$ $\quad \|v\| = 2$ $\quad \|v\| = \sqrt{\langle v, v \rangle}$

$\langle u, w \rangle\ 5$ $\quad \|w\| = 7$

A) $\langle u+v, v+w \rangle =$

$= \langle u, v \rangle + \langle u, w \rangle + \langle v, v \rangle + \langle v, w \rangle$

$= 2 + 5 + 4 + 3 = 14$

B) $\langle 2v+w, 3u+2w \rangle$

$= \langle 2v, 3u \rangle + \langle 2v; 2w \rangle + \langle w + 3u \rangle + \langle w, 2w \rangle$

$= 6 \langle v, u \rangle + 4 \langle v, w \rangle + 3 \langle w, u \rangle + 2 \langle w, w \rangle$

$= 6.2 + 4.3 + 3.5 + 2.49$

$= 137$

C) $\|u+v\| = \sqrt{\langle u+v, u+v \rangle}$

$\langle u+v, u+v \rangle = \sqrt{\langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle}$

$= \sqrt{1 + 2 + 2 + 4} = 3$

Valores próprios e vetores próprios

Definição: Um escalar ($\lambda \in \mathbb{R}$ ou $K \in \mathbb{C}$) é denominado valor próprio de T se existe $v \in V$, $v \neq 0$, tal que $T(v) = \lambda v$

Matriz característica $A - \lambda I_n$

Polinômio característico. $p(\lambda) = \det(A - \lambda I_n)$

Equação característica: $p(\lambda) = 0$

"Os valores próprios são as raízes da equação característica"

Exemplo: Dado o operador linear

$T: \mathbb{R}^2 \to \mathbb{R}^2$ tal que $T(x,y) = (-3x - 2y, 5x + 4y)$

$B = \{(1,0), (0,1)\}$

$T(1,0) = (-3, 5)$ $A = \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$

$T(0,1) = (-2, 4)$

Matriz característica $A - \lambda I_n$

$$\begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\lambda = \begin{bmatrix} -3-\lambda & 5 \\ -2 & 4-\lambda \end{bmatrix}$$

Polinômio característico

$$d(A - \lambda I_n) = (-3-\lambda)(4-\lambda) - (5 \cdot (-2))$$
$$= -12 + 3\lambda - 4\lambda + \lambda^2 + 10$$
$$= \lambda^2 - \lambda - 2 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} \qquad \lambda_1 = 2 \quad \lambda_2 = -1$$

Valores próprios $\lambda_1 = 2$ e $\lambda_2 = -1$

Vetores próprios para $\lambda_1 = 2$

$$\begin{bmatrix} -3-2 & 5 \\ -2 & 4-2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -5x + 5y = 0 \\ -2x + 2y = 0 \end{cases} \qquad x = y$$

Para $x = 1 \to y = 1$

temos $v_1 = (1,1)$

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Ex1) $T: R^2 \to R^2$

$T: (x,y) = (x+y, x-y)$

$T(1,0) = (1,1)$
$T(0,1) = (1,-1)$
$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 1$$

$(1-\lambda)(-1-\lambda) - 1$

$-1-\lambda+\lambda+\lambda^2-1 \Rightarrow \lambda^2 - 2 = 0 \quad \therefore \lambda = \pm\sqrt{2}$

Para $\lambda = \sqrt{2}$

$$\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x - x\sqrt{2} + y = 0 \\ x - y - y\sqrt{2} = 0 \end{cases} \qquad \begin{cases} x(1-\sqrt{2}) + y = 0 \\ x - y(1+\sqrt{2}) = 0 \end{cases}$$

Ex2-) $T: R^2 \to R^2 : T(x,y) = (-y, x)$

$$\begin{cases} T(1,0) = (0,1) \\ T(0,1) = (-1,0) \end{cases} \quad (T) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix}$$

$\lambda^2 + 1 = 0$

$\therefore$ Não existe nem valor e nem vetores próprios

$*\quad T: \mathbb{R}^3 \to \mathbb{R}^3 \quad T(x,y,z) = (x+y,\ x+z,\ y+z)$

$T(1,0,0) = (1,1,0)$
$T(0,1,0) = (1,0,1)$
$T(0,0,1) = (0,1,1)$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Matriz característica $\begin{pmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix}$

Polinômio característico

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} \begin{matrix} 1-\lambda & 1 \\ 1 & -\lambda \\ 0 & 1 \end{matrix} =$$

$-\lambda(1-\lambda)^2 - (1-\lambda) - (1-\lambda) = 0$
$-\lambda(1-2\lambda+\lambda^2) - 1 + \lambda - 1 + \lambda = 0$
$-\lambda(1-2\lambda+\lambda^2) - 2 + 2\lambda = 0$
$-\lambda(1-2\lambda+\lambda^2+2) - 2 = 0$
$-\lambda(\lambda^2 - 2\lambda + 3) - 2 = 0$
$-\lambda^3 + 2\lambda - 3\lambda - 2 = 0$ (Polinômio característico)

d) $T: \mathbb{R}^3 \to \mathbb{R}^3 \quad T(x,y,z) = (y,z,x)$

$T(1,0,0) = (0,0,1)$
$T(0,1,0) = (1,0,0)$
$T(0,0,1) = (0,1,0)$

$$(T) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Matriz característica

$$\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{pmatrix}$$

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$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} \begin{matrix} -\lambda & 1 \\ 0 & -\lambda \\ 1 & 0 \end{matrix} = -\lambda^3 + 1 = 0$$

$$\lambda = \sqrt[3]{1} = 1$$

Para $\lambda = 1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{cases} -x + y = 0 \\ -y + z = 0 \\ x - z = 0 \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} -x + y = 0 \\ -y + z = 0 \end{cases}$$

$y = z = x$ logo $v = (x, y, z)$

$= (x, x, x)$

$= x(1,1,1)$

Todos vetores multiplos de $(1,1,1)$ é um vetor próprio associado ao valor próprio $\lambda = 1$.

e) $T: \mathbb{R}^3 \to \mathbb{R}^3 : T(x,y,z) = (x+y+z, x+y, x+y)$

$T(1,0,0) = (1,1,1)$

$T(0,1,0) = (1,1,1)$

$T(0,0,1) = (1,0,0)$

1) $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$(T) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 2 \end{bmatrix} (A) \qquad A - \lambda I_n$$

$$\begin{vmatrix} -1-\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix} \begin{matrix} -1-\lambda & 1 \\ 0 & -\lambda \\ -1 & 1 \end{matrix} = 0$$

$$-\lambda(-1-\lambda)(2-\lambda) = 0 \quad \therefore \lambda_1 = 0 \quad \lambda_2 = -1 \quad \lambda_3 = 2$$

Para $\lambda_1 = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{cases} -x + y = 0 \\ -x + y + 2z = 0 \end{cases}$$

$x = y$

$-x + y + 2z = 0 \Rightarrow -y + y + 2z = 0 \quad \therefore z = 0$

Para $x = 1$ $y = 1$ $z = 0$

$v = (1,1,0)$

Para $\lambda_1 = -1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y = 0$

$y = 0$

$-x + y + 3z = 0$

$x = 3z$

$v = (3z, 0, z)$

$= z(3,0,1)$

$v = (3,0,1)$

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(Gabarito para o Victor Afonso)

Introd teórica

$$\| f+g \| = \sqrt{\langle f+g, f+g \rangle} = \sqrt{\langle f,f \rangle + \langle f,g \rangle + \langle g,f \rangle + \langle g,g \rangle}$$

$\Rightarrow$ $A = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix}$

a) $A - \lambda I_n$

$$\begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{bmatrix}$$

b) $\det(A - \lambda I_n) = 0$

$$\left| \begin{array}{ccc|cc} -4-\lambda & 1 & 1 & -4-\lambda & 1 \\ 1 & 5-\lambda & -1 & 1 & 5-\lambda \\ 0 & 1 & -3-\lambda & 0 & 1 \end{array} \right. =$$

$$(-4-\lambda)(5-\lambda)(-3-\lambda) + 1 + (-4-\lambda) - (-3-\lambda) = 0$$

$$(-4-\lambda)(5-\lambda)(-3-\lambda) \cancel{+1} \cancel{-4} \cancel{-\lambda} \cancel{+3} \cancel{+\lambda} = 0$$

$$\lambda_1 = -4 \quad \lambda_2 = 5 \quad \lambda_3 = 3$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 9 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\begin{cases} y+z=0 \\ x+9y-z=0 \\ y+z=0 \end{cases}$ $\quad y=-z \quad x-9z-z=0 \quad x=10z$

$v=(x,y,z)$
$=(10z,-z,z)$
$=z(10,-1,1)$

Todo vetor múltiplo do vetor $(10,-1,1)$ é um vetor próprio associado ao valor próprio $\lambda_2=-4$

$\longrightarrow$

$T(1,1,0)=(1,1,1)$
$T(0,-1,1)=(0,-1,-1)$
$T(1,0,2)=(1,-1,0)$

i) $T(x,y,z)$

$(x,y,z)=a(1,1,0)+b(0,-1,1)+c(1,0,2)$

$$\begin{cases} a++c=x \\ a-b=y \\ -b+2c=z \end{cases} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 1 & -1 & 0 & y \\ 0 & -1 & 2 & z \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 0 & 1 & x \\ 0 & -1 & -1 & y-x \\ 0 & 0 & 3 & x-y+z \end{array}\right]$$

$\begin{cases} a++c=x \\ -b-c=-x+y \\ 3c=x-y+z \end{cases}$ $\quad c=\dfrac{x-y+z}{3} \quad a=\dfrac{2x+y-z}{3}$

$b=\dfrac{-x+y-z}{3}+x-y=\dfrac{2x-2y-z}{3}$

/ /

$T(x,y,z) = aT(1,1,0) + bT(0,-1,1) + cT(1,0,2)$

$T(x,y,z) = \frac{2x+y-z}{3}(1,1,1) + \frac{2x-2y-z}{3}(0,-1,-1) + \frac{x-y+z}{3}(1,-1,0)$

Transformação linear Inversa

$F^{-1}(x,y) = (a,b) \therefore F(a,b) = (x,y)$

Exemplo:

$F(1,1) = (1,2)$

$F(2,0) = (4,2)$

$F(x,y) = ?$

$(x,y) = a(1,1) + b(2,0)$

$\begin{cases} a + 2b = x \\ a = y \end{cases} \therefore b = \frac{x-a}{2} = \frac{x-y}{2}$

$(x,y) = a(1,1) + b(2,0)$

$F(x,y) = aF(1,1) + bF(2,0)$

$F(x,y) = y(1,2) + \frac{x-y}{2}(4,2)$

$= (2x-y, x+y)$ $\quad (2a-b, a+b) = (x,y)$

$\begin{cases} 2a-b = x \\ a+b = y \end{cases}$ $\quad 3a = x+y$

$F^{-1}(x,y) = (a,b)$ $\quad a = \frac{x+y}{3}$

$F(a,b) = (x,y)$

$b = \frac{x-2y}{3}$